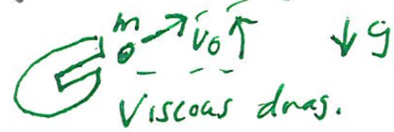
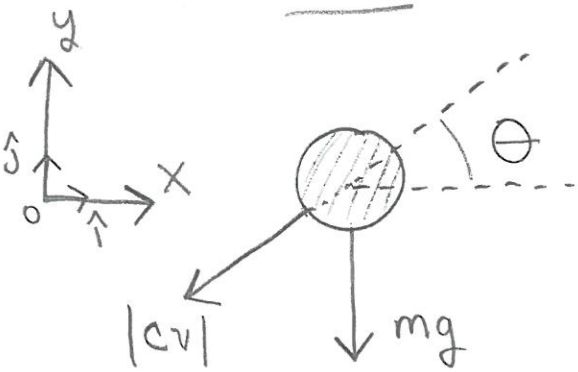
✓+ 2) Cannon Ball (Time to Complete = 2 hours)

Good, But no need for full sentences, Just need barely enough to Know what the Prob. is: Like 

A cannon ball m is launched at angle θ and speed v_0. It is acted on by gravity g and a viscous drag with magnitude $|cv|$.

a) Find position vs. time analytically. ⇒ Find $\vec{r}(t)$.

FBD



LMB

$$\Sigma \vec{F} = m\vec{a}$$

$$-mg\hat{j} + |cv|\left(-\frac{\vec{v}}{|\vec{v}|}\right) = m\vec{a}$$

Since $v = |\vec{v}|$, we can rewrite this as

$$-mg\hat{j} + |c||\vec{v}|\left(-\frac{\vec{v}}{|\vec{v}|}\right) = m\vec{a}$$

$$\Rightarrow \quad -mg\hat{j} - |c|\vec{v} = m\vec{a}$$

Also, since the drag constant c will be taken to be a positive number, we can drop the absolute value signs

Very clear. (& good.) But you could still do At work w/out being quite so expansive (applies throughout). Not bad, just more work than needed.

2) a) (continued)

$$\Rightarrow \quad -mg\,\hat{j} - c\vec{v} = m\vec{a}$$

we can write $\vec{v} = v_x\,\hat{i} + v_y\,\hat{j}$

$$\Rightarrow \quad -mg\,\hat{j} - c(v_x\,\hat{i} + v_y\,\hat{j}) = m\vec{a}$$

$$\vec{a} = \left(-\frac{cv_x}{m}\right)\hat{i} + \left(-g - \frac{cv_y}{m}\right)\hat{j}$$

we can also write $\vec{a} = a_x\,\hat{i} + a_y\,\hat{j}$

$$\Rightarrow \quad a_x\,\hat{i} + a_y\,\hat{j} = \left(-\frac{cv_x}{m}\right)\hat{i} + \left(-g - \frac{cv_y}{m}\right)\hat{j}$$

Now we must equate like components:

$\hat{i}$-direction: $a_x = -\dfrac{cv_x}{m}$

$\hat{j}$-direction: $a_y = -g - \dfrac{cv_y}{m}$

These equations can be solved to get the cannon ball's position as a function of time.

2) a) (continued)

Looking at $\hat{\imath}$-direction equation :

$$a_x = -\frac{c v_x}{m}$$

We know that $a_x = \frac{d v_x}{dt}$

Appendix C.1

$$\Rightarrow \quad \frac{d v_x}{dt} = -\frac{c v_x}{m}$$

← See Simple ODEs box in Raine/Pratap. You can just know that $\dot{z} = -cz \Rightarrow z = C_0 e^{-ct}$

$$\frac{d v_x}{v_x} = \left(-\frac{c}{m}\right) dt$$

Integrating both sides yields

$$\int \frac{d v_x}{v_x} = \int \left(-\frac{c}{m}\right) dt$$

$$\ln(v_x) = \left(-\frac{c}{m}\right) t + C_1$$

To solve for C_1, we have the initial condition that at $t=0$, $v_x = v_0 \cos\theta$

$$\Rightarrow \quad \ln(v_0 \cos\theta) = \left(-\frac{c}{m}\right)(0) + C_1$$

2) a) (continued)

$$\Rightarrow \quad c_1 = \ln(v_0 \cos\theta)$$

we then have $\ln(v_x) = \left(-\frac{c}{m}\right)t + \ln(v_0\cos\theta)$

$$\Rightarrow \quad v_x = e^{\left[\left(-\frac{c}{m}\right)t + \ln(v_0\cos\theta)\right]}$$

$$v_x = \left[e^{\left(-\frac{c}{m}\right)t}\right]\left[e^{\ln(v_0\cos\theta)}\right]$$

$$* \quad v_x = (v_0\cos\theta)\, e^{\left(-\frac{c}{m}\right)t} \quad *$$

We also know that $v_x = \frac{dx}{dt}$

$$\Rightarrow \quad \frac{dx}{dt} = (v_0\cos\theta)\, e^{\left(-\frac{c}{m}\right)t}$$

$$dx = (v_0\cos\theta)\, e^{\left(-\frac{c}{m}\right)t}\, dt$$

Integrating both sides gives

$$\int dx = \int (v_0\cos\theta)\, e^{\left(-\frac{c}{m}\right)t}\, dt$$

$$x = (v_0\cos\theta)\left(-\frac{m}{c}\right)e^{\left(-\frac{c}{m}\right)t} + c_2$$

To solve for c_2, we have the initial condition

at $t = 0$, $x = 0$

2) a) (continued)

$$\Rightarrow \quad (0) = (v_0 \cos\theta)\left(-\frac{m}{c}\right) e^{\left(-\frac{c}{m}\right)(0)} + C_2$$

$$0 = -\frac{m v_0 \cos\theta}{c} + C_2$$

$$C_2 = \frac{m v_0 \cos\theta}{c}$$

Plugging this in yields

$$x = \left(-\frac{m v_0 \cos\theta}{c}\right) e^{\left(-\frac{c}{m}\right)t} + \frac{m v_0 \cos\theta}{c}$$

$$* \quad x = \left(\frac{m v_0 \cos\theta}{c}\right)\left[1 - e^{\left(-\frac{c}{m}\right)t}\right] \quad *$$

We can do the same thing for the $\hat{j}$-direction equation:

$$a_y = -g - \frac{c v_y}{m}$$

2) a) (continued)

$$\frac{dv_y}{dt} = -g - \frac{cv_y}{m}$$

$$\Rightarrow \quad \frac{dv_y}{dt} + \frac{cv_y}{m} = -g$$

First, we must solve the homogenous equation

$$\frac{dv_y}{dt} + \frac{cv_y}{m} = 0$$

The characteristic equation is

$$r + \frac{c}{m} = 0$$

$$\Rightarrow \quad r = -\frac{c}{m}$$

The complementary solution is

$$v_{yc} = C_3 e^{\left(-\frac{c}{m}\right)t}$$

I like this better than the logs you used before.

Since the right-hand side of the nonhomogenous differential equation is a constant, a guess for the particular solution is

$$v_{yp} = C_4$$

$$\Rightarrow \quad \frac{dv_{yp}}{dt} + \frac{cv_{yp}}{m} = -g$$

2) a) (continued)

Plugging in for C_3 we get

$$v_y = \left(v_0 \sin\theta + \frac{mg}{c}\right) e^{\left(-\frac{c}{m}\right)t} - \frac{mg}{c}$$

$$* \quad v_y = (v_0 \sin\theta) e^{\left(-\frac{c}{m}\right)t} + \frac{mg}{c}\left[e^{\left(-\frac{c}{m}\right)t} - 1\right] \quad *$$

we also know that $v_y = \frac{dy}{dt}$

$$\Rightarrow \quad \frac{dy}{dt} = (v_0 \sin\theta) e^{\left(-\frac{c}{m}\right)t} + \frac{mg}{c}\left[e^{\left(-\frac{c}{m}\right)t} - 1\right]$$

$$\int dy = \int \left[(v_0 \sin\theta) e^{\left(-\frac{c}{m}\right)t} + \frac{mg}{c}\left[e^{\left(-\frac{c}{m}\right)t} - 1\right]\right] dt$$

$$y = \left(-\frac{m v_0 \sin\theta}{c}\right) e^{\left(-\frac{c}{m}\right)t} - \left(\frac{m^2 g}{c^2}\right) e^{\left(-\frac{c}{m}\right)t} - \left(\frac{mg}{c}\right)t + C_5$$

To solve for C_5, we have the initial condition at $t=0$, $y=0$

$$\Rightarrow \quad 0 = -\frac{m v_0 \sin\theta}{c} - \frac{m^2 g}{c^2} + C_5$$

$$C_5 = \frac{m v_0 \sin\theta}{c} + \frac{m^2 g}{c^2}$$

Plugging this into the equation for y yields

2) a) (continued)

$$y = -\left(\frac{mv_0\sin\theta}{c} + \frac{m^2 g}{c^2}\right)e^{\left(-\frac{c}{m}\right)t} - \left(\frac{mg}{c}\right)t + \frac{mv_0\sin\theta}{c} + \frac{m^2 g}{c^2}$$

$$* \quad y = \left(\frac{mv_0\sin\theta}{c} + \frac{m^2 g}{c^2}\right)\left[1 - e^{\left(-\frac{c}{m}\right)t}\right] - \left(\frac{mg}{c}\right)t \quad *$$

Thus, the position $\vec{r}(t)$ of the cannon ball from the origin is

$$\vec{r}(t) = x\hat{\imath} + y\hat{\jmath}$$

$$\boxed{\vec{r}(t) = \left(\frac{mv_0\cos\theta}{c}\right)\left[1 - e^{\left(-\frac{c}{m}\right)t}\right]\hat{\imath} + \left[\left(\frac{mv_0\sin\theta}{c} + \frac{m^2 g}{c^2}\right)\left[1 - e^{\left(-\frac{c}{m}\right)t}\right] - \left(\frac{mg}{c}\right)t\right]\hat{\jmath}}$$

✓ looks good

2) b)

Find a numerical solution using $\theta = \pi/4$, $v_0 = 1\,m/s$, $g = 1\,m/s^2$, $m = 1\,Kg$. Use Euler's method.

From part (a) we had

$$-mg\hat{j} - c\vec{v} = m\vec{a}$$

$$* \quad \vec{a} = -g\hat{j} - \left(\frac{c}{m}\right)\vec{v} \quad *$$

If at time t we know $\vec{r}(t)$ = position of cannon ball and $\vec{v}(t)$ = velocity of cannon ball, then we can say that

$$\vec{r}(t+\Delta t) = \vec{r}(t) + \Delta t\,\vec{v}(t)$$
$$\vec{v}(t+\Delta t) = \vec{v}(t) + \Delta t\,\vec{a}(t)$$

✓

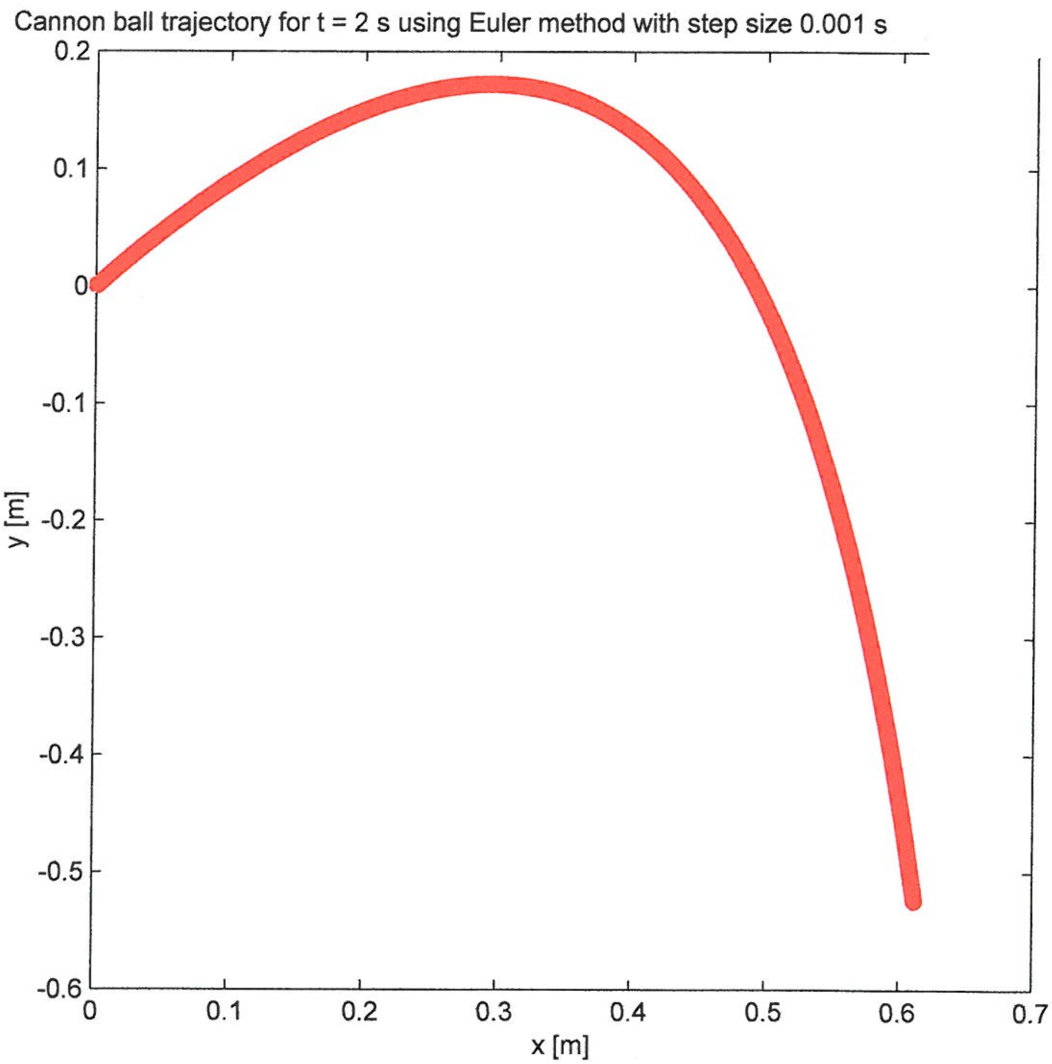
where $\vec{a}(t)$ is given above.

The Matlab program used to solve this problem is attached, as is a graph showing the trajectory for $t = 2$ seconds.

In general you should do such. (assume what you need)

NOTE: I used $c = 1\,Kg/s$ in my numerical solution ✓ since the value of c was not specified.

```
%By
%MAE 5735
%HW 2, Problem 2b

%This program finds a numerical solution to the problem pertaining to the
%motion of a cannon ball acted on by gravity and viscous linear drag.

clc
clear all
close all

%Givens
g = 1; %Graviational acceleration in m/s^2
v0 = 1; %Initial velocity magnitude in m/s
theta = pi/4; %Launch angle in radians
m = 1; %Mass in kg

%Select value for air drag constant
c = 1; %kg/s

%Initial vectors of interest
r = [0 0]'; %Position vector
v = [v0*cos(theta) v0*sin(theta)]'; %Velocity vector

%Implementation of Euler's method
step = 0.001; %Step size in seconds used in the Euler method
time_span = 2; %Total time in seconds used to calculate trajectory
for k = 1:(time_span/step)
    a = (-c/m)*v-[0 g]'; %Acceleration
    r = r+(step)*v;
    v = v+(step)*a;
    plot(r(1),r(2),'ro')
    hold on
end
axis square
xlabel('x [m]')
ylabel('y [m]')
title('Cannon ball trajectory for t = 2 s using Euler method with step size 0.001 s
(by , )
```


Cannon ball trajectory for t = 2 s using Euler method with step size 0.001 s
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
y [m]
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
x [m]

2) c)

Compare the numeric and analytic solutions. At $t = 2$ how big is the error? How does the error depend on step size?

Using the Matlab code from part (b), the position of the cannon ball at $t = 2$ is

$x = 0.6115$ meters

$y = -0.5237$ meters

$$\Rightarrow \boxed{\begin{array}{l}\vec{r} = (0.6115\ m)\hat{i} - (0.5237\ m)\hat{j} \\ \text{(at } t = 2\text{ s, numeric result)}\end{array}}$$

The analytic result, from part (a), was

$$\vec{r}(t) = \left(\frac{m v_0 \cos\theta}{c}\right)\left[1 - e^{\left(-\frac{c}{m}\right)t}\right]\hat{i} + \left[\left(\frac{m v_0 \sin\theta}{c} + \frac{m^2 g}{c^2}\right)\left[1 - e^{\left(-\frac{c}{m}\right)t}\right] - \left(\frac{mg}{c}\right)t\right]\hat{j}$$

with $m = 1\text{ kg}$, $v_0 = 1\text{ m/s}$, $\theta = \pi/4$, $c = 1\text{ kg/s}$, and $t = 2\text{ s}$ we get

$$\boxed{\begin{array}{l}\vec{r} = (0.6114\ m)\hat{i} - (0.5239\ m)\hat{j} \\ \text{(at } t = 2\text{ s, analytic result)}\end{array}}$$

2) c) (continued)

$\Rightarrow$ At $t = 2\,s$, the error between the analytic and numeric solutions is

$$\overrightarrow{error} = \vec{r}_{numeric} - \vec{r}_{analytic}$$

$$\overrightarrow{error} = \left[(0.6115\text{ m})\hat{i} - (0.5237\text{m})\hat{j}\right] - \left[(0.6114\text{ m})\hat{i} - (0.5239\text{m})\hat{j}\right]$$

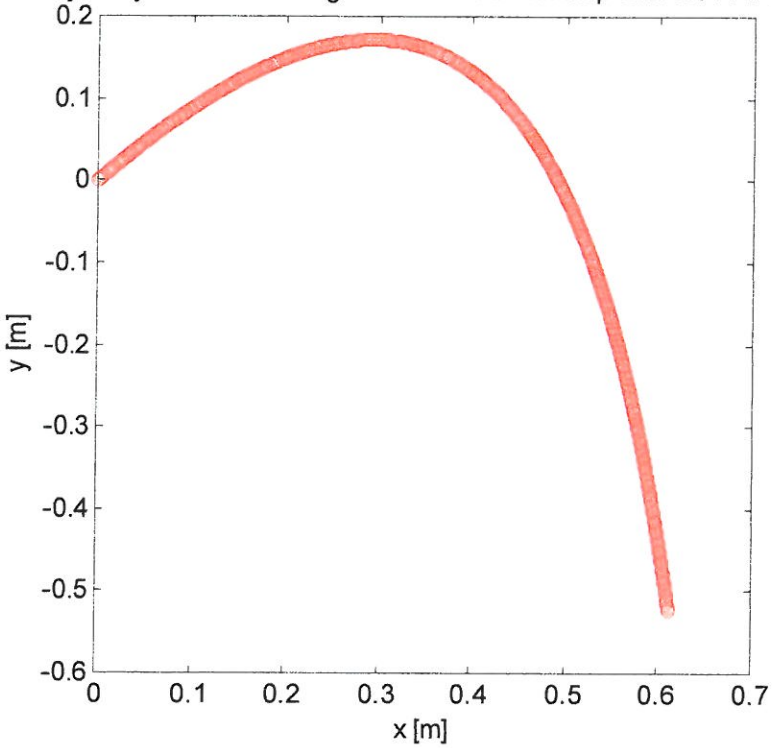
$$\boxed{\overrightarrow{error} = (0.0001\text{ m})\hat{i} + (0.0002\text{ m})\hat{j}}$$

See attached plot of the analytic solution vs. the numeric solution with a step size of 0.001 s for $t = 2\,s$. The Matlab code used to generate these plots is also attached.

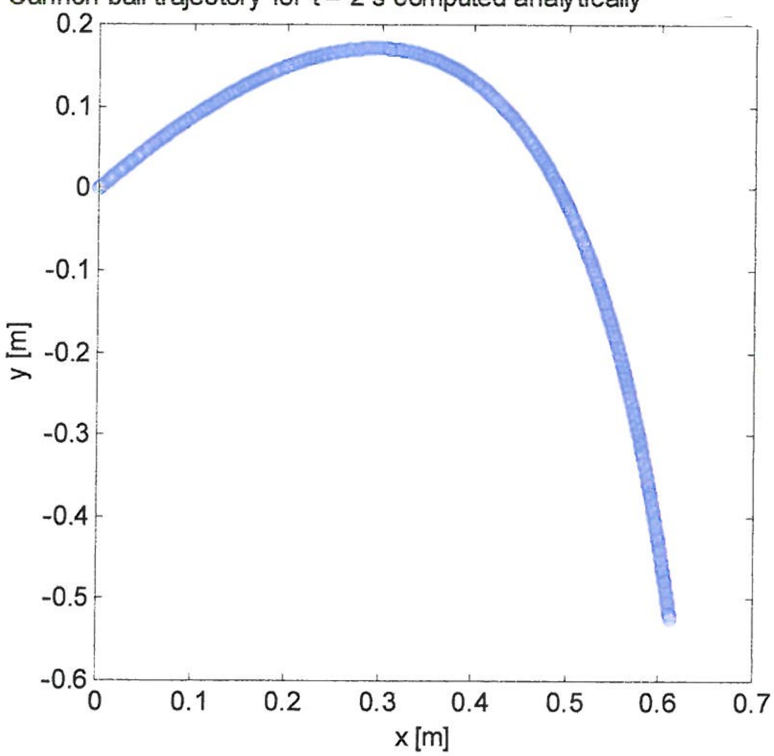
```
%
%MAE 5735
%HW 2, Problem 2c

%This program finds a numerical solution to the problem pertaining to the
%motion of a cannon ball acted on by gravity and viscous linear drag.

clc
clear all
close all

%Givens
g = 1; %Graviational acceleration in m/s^2
v0 = 1; %Initial velocity magnitude in m/s
theta = pi/4; %Launch angle in radians
m = 1; %Mass in kg

%Select value for air drag constant
c = 1; %kg/s

%Initial vectors of interest
r = [0 0]'; %Position vector
v = [v0*cos(theta) v0*sin(theta)]'; %Velocity vector

%Implementation of Euler's method
step = 0.001; %Step size in seconds used in the Euler method
time_span = 2; %Total time in seconds used to calculate trajectory
for k = 1:(time_span/step)
    a = (-c/m)*v-[0 g]'; %Acceleration
    r = r+(step)*v;
    v = v+(step)*a;
    plot(r(1),r(2),'ro')
    hold on
end
axis square
xlabel('x [m]')
ylabel('y [m]')
title('Cannon ball trajectory for t = 2 s using Euler method with step size 0.001 s
(by A.Savas)')

%The following plots the cannon ball trajectory from the analytical
%solution for t = 2 s.
figure
for t = 0:step:2
    position = [((m*v0*cos(theta))/c)*(1-exp((-c/m)*t)) (((m*v0*sin(theta))/c)+(m^2*g)
/(c^2))*(1-exp((-c/m)*t))-((m*g)/c)*t]';
    plot(position(1), position(2),'bo')
    hold on
end
axis square
```

```
xlabel('x [m]')
ylabel('y [m]')
title('Cannon ball trajectory for t = 2 s computed analytically                 )
```

Cannon ball trajectory for t = 2 s using Euler method with step size 0.001 s



Cannon ball trajectory for t = 2 s computed analytically



2) c) (continued)

also called "method" or "truncation"

The error depends on step size. Two prominent errors are the discretization error and the roundoff error. The discretization error decreases as the step size is decreased because finer and finer time steps are being used to calculate the solution. This keeps overestimation to a minimum. However, as the step size is decreased the roundoff error increases. This is because roundoff errors are accumulated at each time step and as the step size is decreased the number of time steps increases. The discretization error is usually on the order of the step size to some power n (in this case 1 = n) and the roundoff error goes as $\sqrt{n}\,\epsilon$ where n is the number of time steps and ϵ is the roundoff error per calculation

Why?

$\Rightarrow$ If t = step size, then $n = \frac{\text{timespan}}{t}$ and we have

$$\boxed{\text{Error} \approx t + \sqrt{\frac{(\text{timespan})}{t}}\,\epsilon}$$

2) c) (continued)

Thus, there is a point when the increase in roundoff error from lowering the step size will overtake the corresponding decrease in discretization error. At this point, decreasing the step size further will lead to more overall error.

A plot showing some of this would be a + (plus).

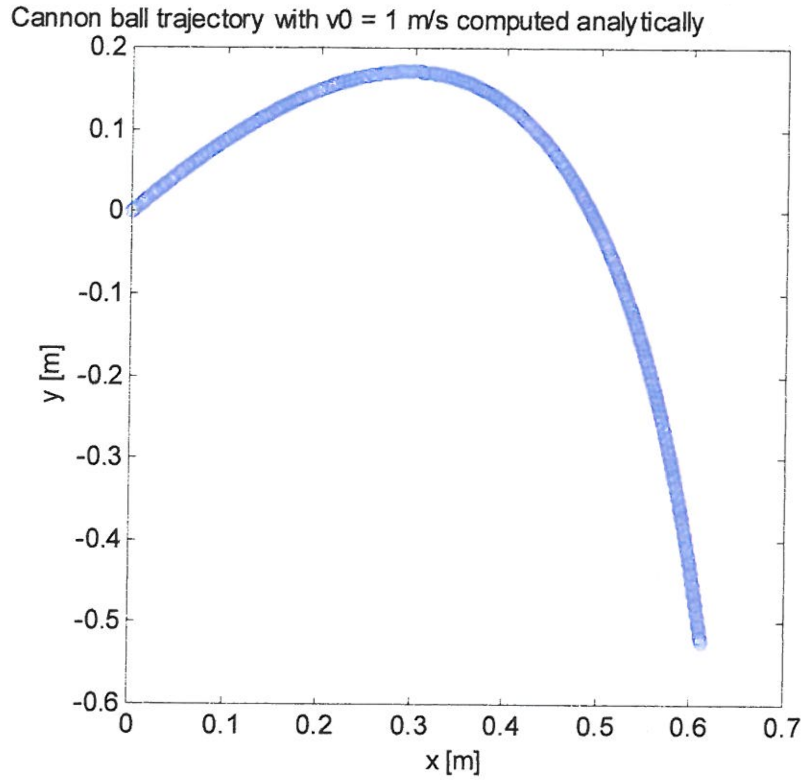
2) d)

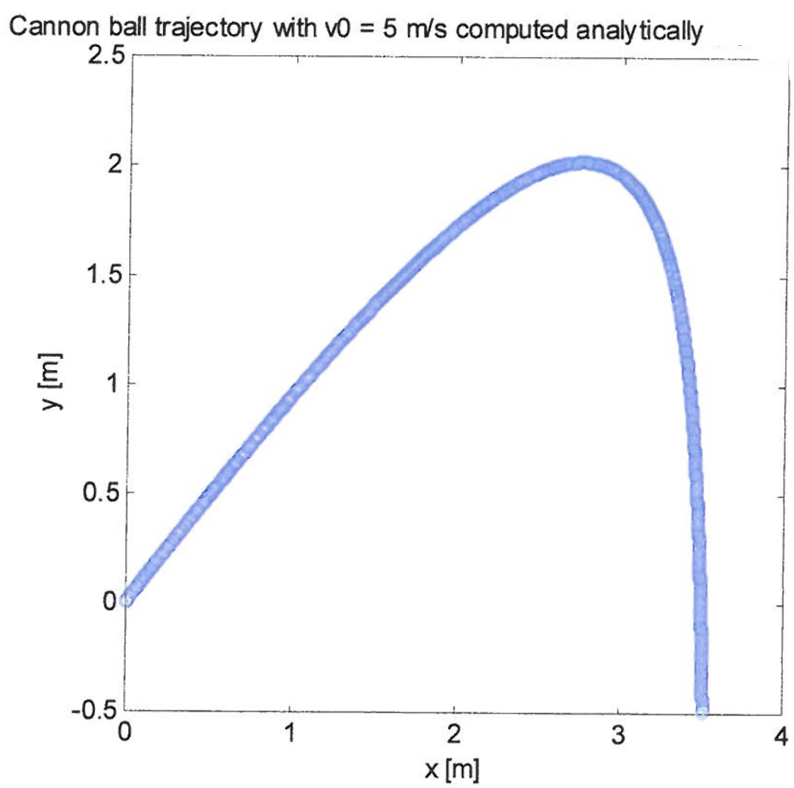
Use larger and larger values of v_0 and for each trajectory choose a time interval so the cannon at least gets back to the ground. Plot the trajectories (using equal scale for the x and y axis). As $v \to \infty$ what is the eventual shape?

See the attached Matlab code and plots

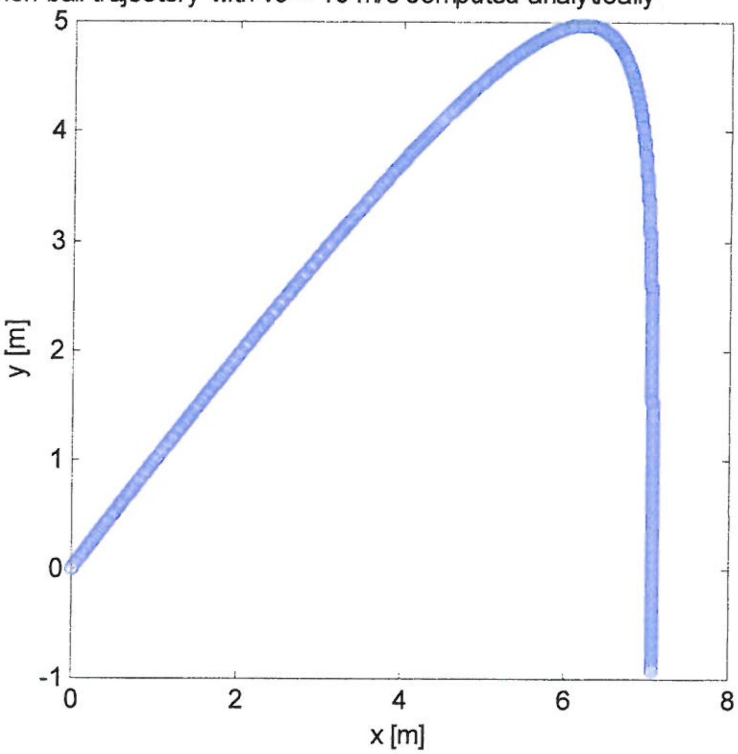
The attached Matlab code was used to plot the various trajectories by adjusting the values of v_0 and the overall timespan traversed by the "for-loop" such that the cannon ball at least makes it back to the ground.

```
%By
%MAE 5735
%HW 2, Problem 2d

%This program uses the analytic solution to the problem pertaining to the
%motion of a cannon ball acted on by gravity and viscous linear drag to
%plot trajectories for various initial speeds, v0.

clc
clear all
close all

%Givens
g = 1; %Graviational acceleration in m/s^2
v0 = 1; %Initial velocity magnitude in m/s
theta = pi/4; %Launch angle in radians
m = 1; %Mass in kg

%Select value for air drag constant
c = 1; %kg/s

%The following plots the cannon ball trajectory from the analytical
%solution for a certain time interval.
step = 0.001; %Step size for plotting analytic solution
for t = 0:step:2
    position = [((m*v0*cos(theta))/c)*(1-exp((-c/m)*t)) (((m*v0*sin(theta))/c)+(m^2*g)
/(c^2))*(1-exp((-c/m)*t))-((m*g)/c)*t]';
    plot(position(1), position(2),'bo')
    hold on
end
axis square
xlabel('x [m]')
ylabel('y [m]')
title('Cannon ball trajectory with v0 = 1 m/s computed analytically
    ')
```

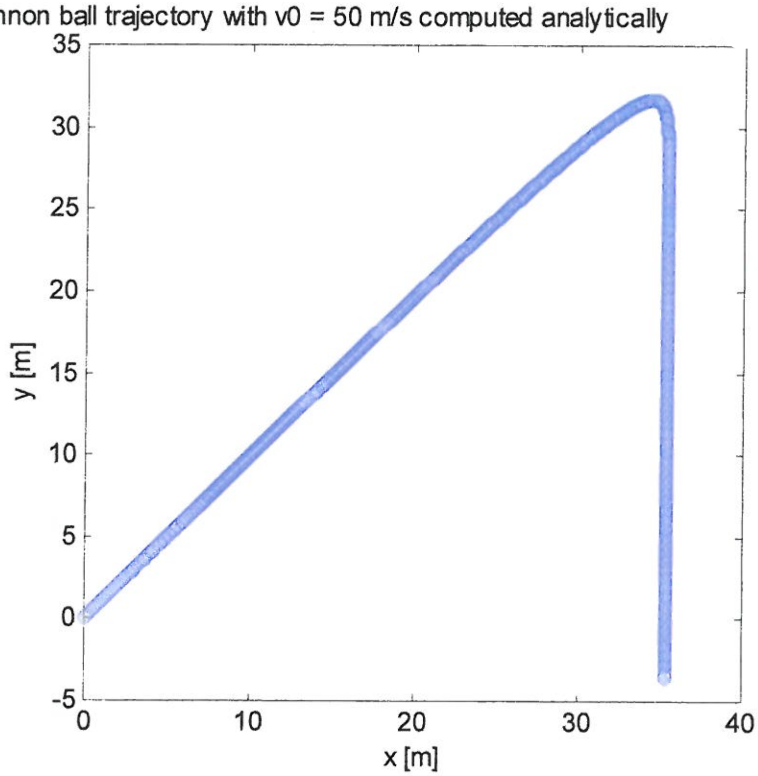

Cannon ball trajectory with v0 = 1 m/s computed analytically
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
y [m]
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
x [m]


Cannon ball trajectory with v0 = 5 m/s computed analytically
2.5
2
1.5
1
0.5
0
-0.5
y [m]
0
1
2
3
4
x [m]

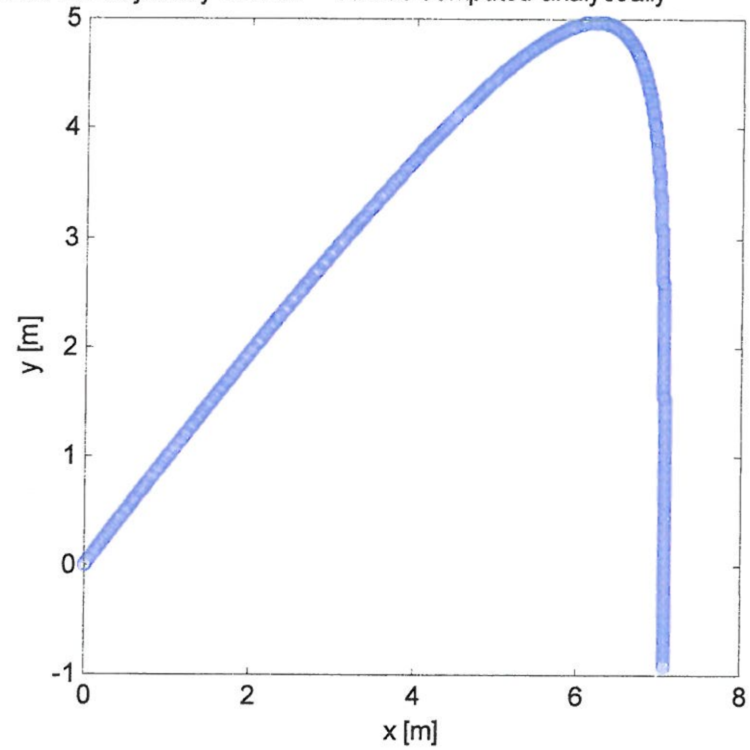
Cannon ball trajectory with v0 = 10 m/s computed analytically



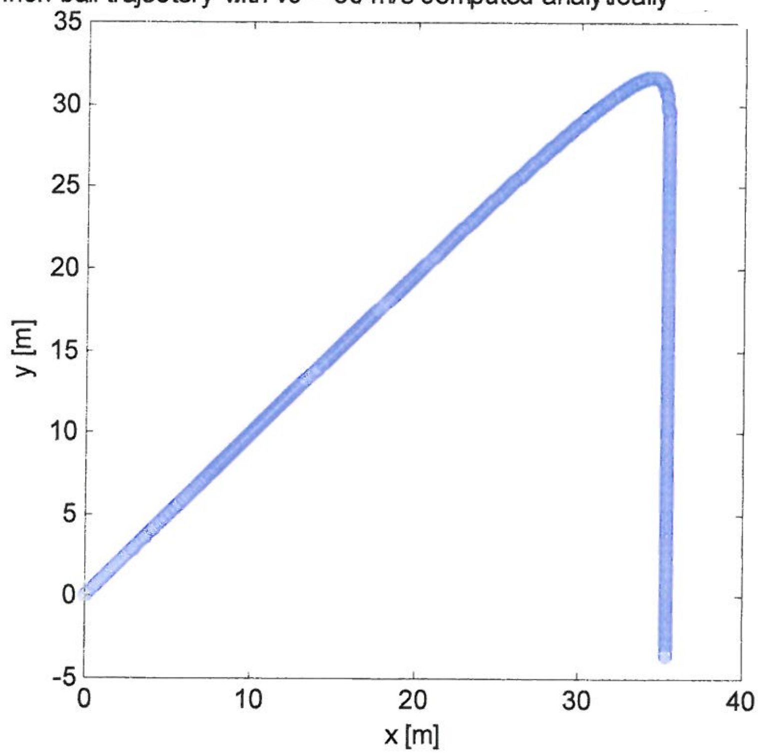
Cannon ball trajectory with v0 = 50 m/s computed analytically

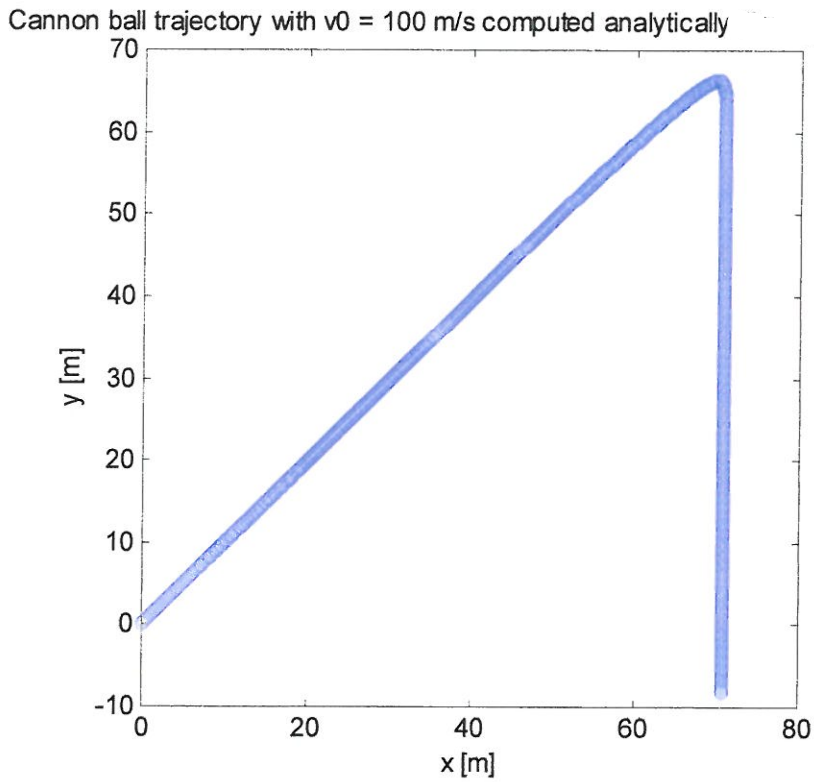


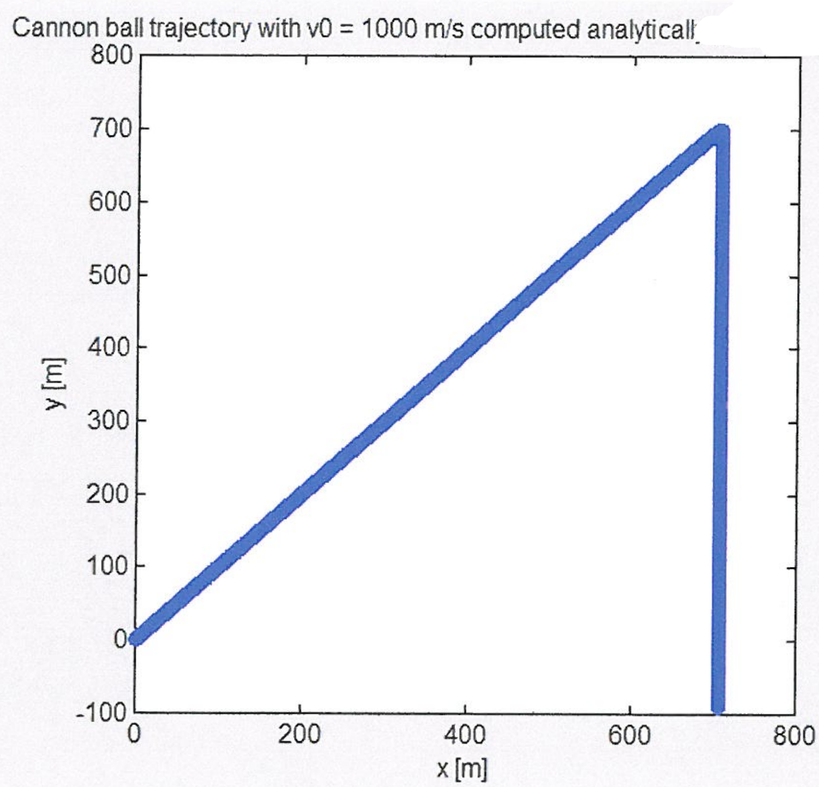
Cannon ball trajectory with v0 = 10 m/s computed analytically



Cannon ball trajectory with v0 = 50 m/s computed analytically



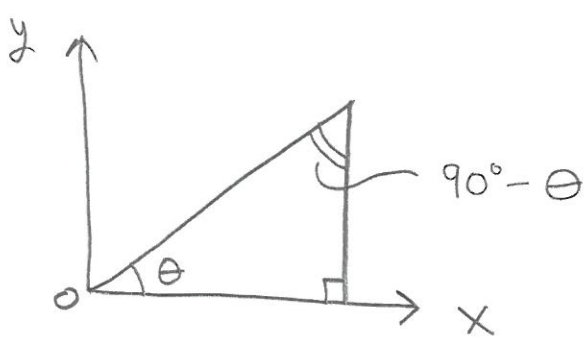

Cannon ball trajectory with v0 = 100 m/s computed analytically
70
60
50
40
30
20
10
0
-10
y [m]
0
20
40
60
80
x [m]


Cannon ball trajectory with v0 = 1000 m/s computed analyticall
800
700
600
500
400
300
200
100
0
-100
y [m]
0
200
400
600
800
x [m]

√+

2) d) (continued)

As can be seen from the previous plots, as $v_0 \to \infty$ the eventual shape is part of a right triangle. More specifically, the trajectory traces out the hypotenuse and vertical leg of a right triangle. In other words, the trajectory consists of a straight slanted section angled to the horizontal at the launch angle and then a sharp turn at an angle ($90°$ - launch angle) leading to a straight downward trajectory. An example trajectory for $v_0 \to \infty$ is shown below:



Note: Max range results on later pages follow from this picture

where θ is the launch angle

2) e)

For any given v_0 there is a best launch angle θ^* for maximizing the range. As $v_0 \to \infty$ to what angle does θ^* tend?

The analytical equation from part (a) for $\vec{r}(t)$ was

$$\vec{r}(t) = \left(\frac{mv_0\cos\theta}{c}\right)\left[1-e^{\left(-\frac{c}{m}\right)t}\right]\hat{i} + \left[\left(\frac{mv_0\sin\theta}{c} + \frac{m^2 g}{c^2}\right)\left[1-e^{\left(-\frac{c}{m}\right)t}\right] - \left(\frac{mg}{c}\right)t\right]\hat{j}$$

To maximize the range we want to maximize the x-componet of $\vec{r}(t)$

$$\Rightarrow \quad r_x = \left(\frac{mv_0\cos\theta}{c}\right)\left[1-e^{\left(-\frac{c}{m}\right)t}\right] \quad \Rightarrow \quad \frac{dr_x}{d\theta} = 0$$

We need to know the time it takes for the cannon ball to hit the ground, assuming it starts from $y=0$.

$\Rightarrow$ Need time at which $r_y = 0$

$$\Rightarrow \left(\frac{mv_0\sin\theta}{c} + \frac{m^2 g}{c^2}\right)\left[1-e^{\left(-\frac{c}{m}\right)t}\right] - \left(\frac{mg}{c}\right)t = 0$$

2) e) (continued)

$$\Rightarrow \left(\frac{m v_0 \sin\theta}{c} + \frac{m^2 g}{c^2}\right)\left[1 - e^{\left(-\frac{c}{m}\right)t}\right] = \left(\frac{mg}{c}\right)t$$

$$1 - e^{\left(-\frac{c}{m}\right)t} = \frac{\left(\frac{mg}{c}\right)t}{\left(\frac{m v_0 \sin\theta}{c} + \frac{m^2 g}{c^2}\right)}$$

$\Rightarrow$ This is very difficult to solve explicitly for t

← impossible analytically?

$\Rightarrow$ Will determine θ^* numerically rather than analytically.

See the attached Matlab program. This program determine the launch angle, θ, which achieves the largest range for various initial speeds, V_0.

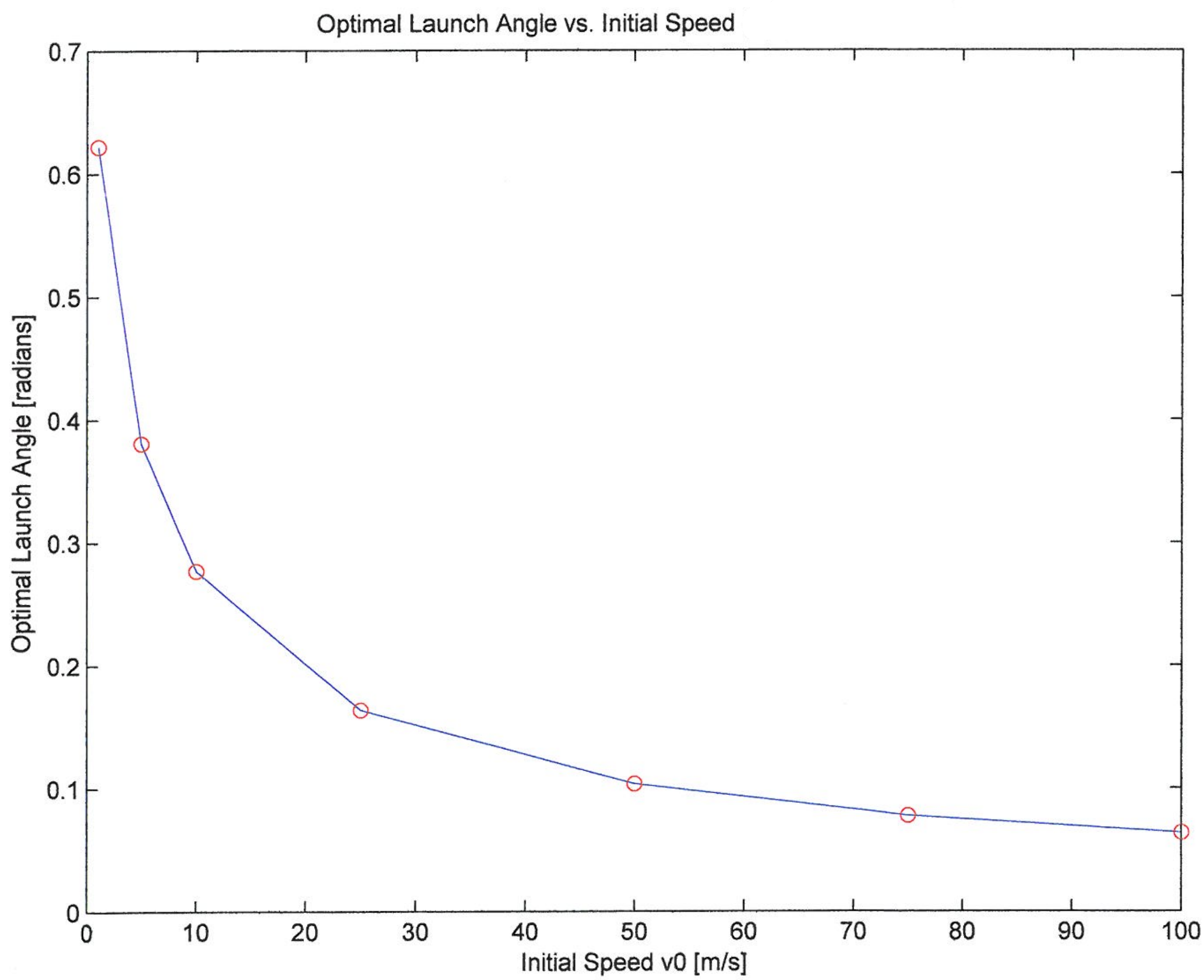
See the attached plot as well.

```
%By
%MAE 5735
%HW 2, Problem 2e

%This program uses the analytic solution to the problem pertaining to the
%motion of a cannon ball acted on by gravity and viscous linear drag to
%determine the optimal launch angle, theta, which gives rise to the maximum
%achievable range based on a given initial speed, v0.

clc
clear all
close all

%Givens
g = 1; %Graviational acceleration in m/s^2
v0 = 1; %Initial velocity magnitude in m/s
m = 1; %Mass in kg

%Select value for air drag constant
c = 1; %kg/s

%The following determines the optimal launch angle in radians which, for a
%given v0, gives the largest range.
step = 0.001; %Step size for stepping through analytic solution
max_range = 0; %Set arbitrary maximum range
for theta = 0.001:.001:1.5 %Try out various launch angles
        for t = 0:step:800
            position = [((m*v0*cos(theta))/c)*(1-exp((-c/m)*t)) (((m*v0*sin(theta))/c)+(m^2*g)/(c^2))*(1-exp((-c/m)*t))-((m*g)/c)*t]';
            if position(2)<0 %Checks if cannon ball reaches the ground
                if position(1) >= max_range; %Checks if current theta gives a new max range
                    max_range = position(1); %Updates the max range
                    optimal_theta = theta; %Updates the optimal theta
                end
                break %Stops stepping through time if the cannon ball reaches the ground
            end
        end
end
optimal_theta %Displays the optimal theta in the command window
```

Various other methods are:

1) Root finding

2) Using 'events' in ODE23 or ODE45


Optimal Launch Angle vs. Initial Speed
Optimal Launch Angle [radians]
Initial Speed v0 [m/s]
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0
10
20
30
40
50
60
70
80
90
100

2) e) (continued)

The results from the Matlab code (shown in the previous plot) are:

Initial Speed V_0 [m/s]	Optimal Launch Angle [rad]
1	0.622
5	0.381
10	0.277
25	0.164
50	0.104
75	0.078
100	0.064

Thus, it is evident from these results and from the previous plot that as $V_0 \rightarrow \infty$, the optimal launch angle tends to zero radians

$$\Rightarrow \boxed{\theta^* \rightarrow 0 \text{ radians}}$$

✓ +