(Time to complete = 1 hour)

4) What means "rate of change of angular momentum"? Consider a moving point C (moving relative to a Newtonian frame F that has an origin O). For which of these definitions of $\vec{H}_{/C}$ is the following equation of motion true (that is, consistent with $\vec{F} = m\vec{a}$)?

$$\vec{M}_C = \dot{\vec{H}}_{/C}$$

a) $\vec{H}_{/C} = \vec{r}_{P/C'} \times \vec{v}_{P/O}$

where C' is a point fixed in F that instantaneously coincides with C

$$\Rightarrow \dot{\vec{H}}_{/C} = (\dot{\vec{r}}_{P/C'} \times \vec{v}_{P/O}) + (\vec{r}_{P/C'} \times \dot{\vec{v}}_{P/O})$$

$$\dot{\vec{H}}_{/C} = (\vec{v}_{P/C'} \times \vec{v}_{P/O}) + (\vec{r}_{P/C'} \times \vec{a}_{P/O})$$

Since C' is fixed in F, its velocity is zero. The origin of F is also not moving (is fixed). Since C' and O are both fixed, the velocity of particle P with respect to C' and O will be the same

$$\Rightarrow \vec{v}_{P/C'} = \vec{v}_{P/O}$$ ✓

4) a) (continued)

$$\Rightarrow \quad \dot{\vec{H}}_{/c} = (\cancel{\vec{v}_{p/o} \times \vec{v}_{p/o}})^{\,0} + (\vec{r}_{p/c'} \times \vec{a}_{p/o})$$

$$\dot{\vec{H}}_{/c} = \vec{r}_{p/c'} \times \vec{a}_{p/o}$$

we also have the condition that c' instantaneously coincides with c

$$\Rightarrow \quad \vec{r}_{p/c'} = \vec{r}_{p/c}$$

we then have

$$\dot{\vec{H}}_{/c} = \vec{r}_{p/c} \times \vec{a}_{p/o}$$

Finally, since the Newtonian frame F has an origin O, we know that $\vec{a}_{p/o} = \vec{a}_{p/F}$

$$\Rightarrow \quad \boxed{\dot{\vec{H}}_{/c} = \vec{r}_{p/c} \times \vec{a}_{p/F}}$$

$\Rightarrow$ It has been shown that this definition works in general

✓

4) b)

$$\vec{H}_{/c} = \vec{r}_{p/c} \times \vec{v}_{p/o}$$

Since c is a moving point, we can write

$$\vec{v}_{p/o} = \vec{v}_{p/c} + \vec{v}_{c/o}$$

$$\Rightarrow \quad \vec{H}_{/c} = \vec{r}_{p/c} \times (\vec{v}_{p/c} + \vec{v}_{c/o})$$

$$\vec{H}_{/c} = (\vec{r}_{p/c} \times \vec{v}_{p/c}) + (\vec{r}_{p/c} \times \vec{v}_{c/o})$$

Taking the time derivative, we get

$$\dot{\vec{H}}_{/c} = (\dot{\vec{r}}_{p/c} \times \vec{v}_{p/c}) + (\vec{r}_{p/c} \times \dot{\vec{v}}_{p/c}) + (\dot{\vec{r}}_{p/c} \times \vec{v}_{c/o}) + (\vec{r}_{p/c} \times \dot{\vec{v}}_{c/o})$$

$$\dot{\vec{H}}_{/c} = \cancelto{0}{(\vec{v}_{p/c} \times \vec{v}_{p/c})} + (\vec{r}_{p/c} \times \vec{a}_{p/c}) + (\vec{v}_{p/c} \times \vec{v}_{c/o}) + (\vec{r}_{p/c} \times \vec{a}_{c/o})$$

$$\Rightarrow \quad \dot{\vec{H}}_{/c} = (\vec{r}_{p/c} \times \vec{a}_{p/c}) + (\vec{v}_{p/c} \times \vec{v}_{c/o}) + (\vec{r}_{p/c} \times \vec{a}_{c/o})$$

We also know that

$$\vec{a}_{p/o} = \vec{a}_{p/c} + \vec{a}_{c/o}$$

4) b) (continued)

Thus, we can write

$$(\vec{r}_{P/C} \times \vec{a}_{P/C}) + (\vec{r}_{P/C} \times \vec{a}_{C/O})$$

$$= \left[\vec{r}_{P/C} \times (\vec{a}_{P/C} + \vec{a}_{C/O})\right]$$

$$= \vec{r}_{P/C} \times \vec{a}_{P/O}$$

Plugging this into the original expression yields

$$\dot{\vec{H}}_{/C} = (\vec{r}_{P/C} \times \vec{a}_{P/O}) + (\vec{v}_{P/C} \times \vec{v}_{C/O})$$

Again, since O is the origin of the Newtonian frame F, $\vec{a}_{P/O} = \vec{a}_{P/F}$

$$\Rightarrow \boxed{\dot{\vec{H}}_{/C} = (\vec{r}_{P/C} \times \vec{a}_{P/F}) + (\vec{v}_{P/C} \times \vec{v}_{C/O})}$$

In order for our expression for $\dot{\vec{H}}_{/C}$ to equal what we need, $(\vec{v}_{P/C} \times \vec{v}_{C/O})$ must equal 0.

4) b) (continued)

$(\vec{v}_{p|c} \times \vec{v}_{c|o}) = 0$ if the following scenarios are true:

$\vec{v}_{p|c} = 0$

$\Rightarrow$ The particle P and the point C are moving at the same velocity with respect to the origin of the Newtonian frame F

$\vec{v}_{c|o} = 0$

$\Rightarrow$ The point C is stationary with respect to the origin of the Newtonian frame F (can only happen instantaneously) ✓

$\vec{v}_{p|c} \times \vec{v}_{c|o} = 0$

$\Rightarrow$ The particle P is moving in the same direction with respect to point C as the point C is moving away from the origin of the Newtonian frame F or in the opposite direction.

$\Rightarrow$ Motion of particle P with respect to the point C is in-line with point C's motion with respect to the origin of the Newtonian frame F.

4) b) (continued)

$\Rightarrow$ $\boxed{\text{Definition works for some special cases concerning the motions of P and C that I previously specified.}}$

4) c)

$$\vec{H}_{/C} = \vec{r}_{P/C} \times \vec{v}_{P/C}$$

Again, we have $\vec{v}_{P/O} = \vec{v}_{P/C} + \vec{v}_{C/O}$

$$\Rightarrow \quad \vec{v}_{P/C} = \vec{v}_{P/O} - \vec{v}_{C/O}$$

Plugging this in, we get

$$\vec{H}_{/C} = \vec{r}_{P/C} \times (\vec{v}_{P/O} - \vec{v}_{C/O})$$

$$\vec{H}_{/C} = (\vec{r}_{P/C} \times \vec{v}_{P/O}) - (\vec{r}_{P/C} \times \vec{v}_{C/O})$$

$\Rightarrow$ Differentiating with respect to time gives

$$\dot{\vec{H}}_{/C} = (\dot{\vec{r}}_{P/C} \times \vec{v}_{P/O}) + (\vec{r}_{P/C} \times \dot{\vec{v}}_{P/O}) - \left[(\dot{\vec{r}}_{P/C} \times \vec{v}_{C/O}) + (\vec{r}_{P/C} \times \dot{\vec{v}}_{C/O})\right]$$

$$\dot{\vec{H}}_{/C} = (\vec{v}_{P/C} \times \vec{v}_{P/O}) + (\vec{r}_{P/C} \times \vec{a}_{P/O}) - \left[(\vec{v}_{P/C} \times \vec{v}_{C/O}) + (\vec{r}_{P/C} \times \vec{a}_{C/O})\right]$$

4) c) (continued)

We know that $\vec{v}_{P/O} = \vec{v}_{P/C} + \vec{v}_{C/O}$

$$\Rightarrow \vec{v}_{P/C} = \vec{v}_{P/O} - \vec{v}_{C/O}$$

We can rewrite the original expression as

$$\dot{\vec{H}}_{/C} = (\vec{v}_{P/C} \times \vec{v}_{P/O}) - (\vec{v}_{P/C} \times \vec{v}_{C/O}) + (\vec{r}_{P/C} \times \vec{a}_{P/O}) - (\vec{r}_{P/C} \times \vec{a}_{C/O})$$

$$\dot{\vec{H}}_{/C} = \vec{v}_{P/C} \times (\vec{v}_{P/O} - \vec{v}_{C/O}) + (\vec{r}_{P/C} \times \vec{a}_{P/O}) - (\vec{r}_{P/C} \times \vec{a}_{C/O})$$

$$\dot{\vec{H}}_{/C} = \cancelto{0}{\vec{v}_{P/C} \times (\vec{v}_{P/C})} + (\vec{r}_{P/C} \times \vec{a}_{P/O}) - (\vec{r}_{P/C} \times \vec{a}_{C/O})$$

$$\dot{\vec{H}}_{/C} = (\vec{r}_{P/C} \times \vec{a}_{P/O}) - (\vec{r}_{P/C} \times \vec{a}_{C/O})$$

Again, since O is the origin of the Newtonian frame F, we can write $\vec{a}_{P/O} = \vec{a}_{P/F}$

4) c) (continued)

$$\Rightarrow \boxed{\dot{\vec{H}}_{/C} = (\vec{r}_{P/C} \times \vec{a}_{P/F}) - (\vec{r}_{P/C} \times \vec{a}_{C/O})}$$

In order for this expression to equal what we want, we need $(\vec{r}_{P/C} \times \vec{a}_{C/O}) = 0$

This is true if

$\vec{r}_{P/C} = 0$ ✓

$\Rightarrow$ Particle P and the point C are always at the same point in space with respect to the origin of the Newtonian frame F

$\vec{a}_{C/O} = 0$ ✓

$\Rightarrow$ The point C is not accelerating with respect to the origin O

$(\vec{r}_{P/C} \times \vec{a}_{C/O}) = 0$ ✓

$\Rightarrow$ The acceleration of point C with respect to the origin O is in-line with the position vector from point C to particle P.

4) c) (continued)

$\Rightarrow$ $\boxed{\text{Definition works for some special cases concerning the motions of P and C that I previously specified.}}$