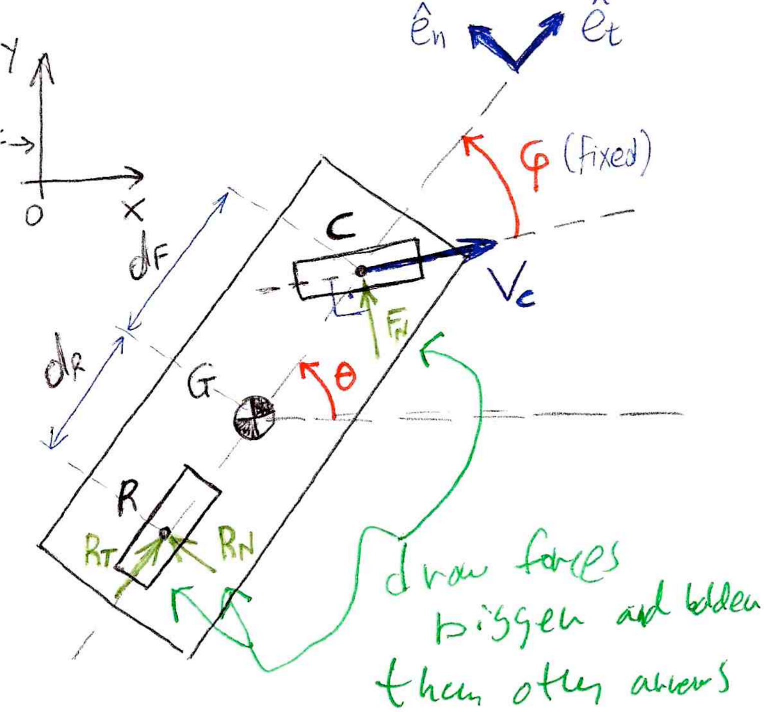
Problem #16: Car with rear brakes locked

When I solved this, the question said "fixed steering", not "straight fixed steering". Thus I will present the $\varphi = \varphi_0 \neq 0$ case and then substitute $\varphi = 0$.

First of all, we make the assumption/simplification that each pair of wheels can be substituted by one wheel ~~at~~ on the longitudinal axis of the car:



$d = d_R + d_F$: car's wheelbase

F_N: normal force on rolling wheel

R_N: —"— friction force on braking (rear) wheel, along $\hat{e}_n$.

R_T: friction force along $\hat{e}_t$.

R: (point) rear wheel

C: (point) front wheel

AMB/C: $\sum \vec{M}_{/C} = \vec{H}_{/C} \Rightarrow F_N \cdot 0 + R_T \cdot 0 + \boxed{(-d\hat{e}_t \times R_N \hat{e}_n) = }$

$$\boxed{= \vec{r}_{G/C} \times m\vec{a}_G + I^G \cdot \ddot{\theta}\hat{k}} \quad (*)$$

Terms one by one:

$\vec{r}_{G/C} = -d_F \cdot \hat{e}_t$, $\vec{\omega} = \dot{\theta}\hat{k}$, $\longrightarrow$

$$\ldots dt\, v_G = \frac{d}{dt}\left(V_{G/C} + V_C\right) = \frac{d}{dt}\left(\vec{\omega}\times\vec{r}_{G/C} + \vec{V}_{/C}\right).$$

$$\vec{V}_C = V_C\cdot\hat{e}_t + \vec{V}_C\cdot\hat{e}_n = V_C\cdot\cos\varphi\,\hat{e}_t + V_C\cdot\sin(-\varphi)\,\hat{e}_n \Rightarrow$$

$$\vec{V}_C = V_C\cdot\cos\varphi\,\hat{e}_t - V_C\cdot\sin\varphi\cdot\hat{e}_n.$$

Recall:

$$\dot{\hat{e}}_t = \dot{\theta}\,\hat{e}_n$$

$$\dot{\hat{e}}_n = -\dot{\theta}\,\hat{e}_t$$

$$\vec{a}_G \overset{\vec{V}_C}{=\!=} \frac{d}{dt}\left(\dot{\theta}\hat{k}\times\vec{r}_{G/C} + V_C\left(\cos\varphi\,\hat{e}_t - \sin\varphi\,\hat{e}_n\right)\right) =$$

$$= \frac{d}{dt}\left(\dot{\theta}\hat{k}\times(-d_F)\hat{e}_t\right) + \frac{d}{dt}\left(V_C\cos\varphi\,\hat{e}_t - V_C\sin\varphi\,\hat{e}_n\right) \xRightarrow{\varphi=\text{constant}}$$

$$a_G = \left[(\ddot{\theta}\hat{k}\times(-d)\hat{e}_t + \dot{\theta}\hat{k}\times(-d_F)\dot{\hat{e}}_t\right] +$$

$$+\left[\dot{V}_C\cos\varphi\,\hat{e}_t + V_C\cdot\cos\varphi\,\dot{\hat{e}}_t - \dot{V}_C\sin\varphi\,\hat{e}_n - V_C\cdot\sin\varphi\,\hat{e}_n\right] =$$

$$\left[-\ddot{\theta}d_F\hat{e}_n + \dot{\theta}\hat{k}\times(-d_F)\dot{\theta}\hat{e}_n\right] +$$

$$\left[\dot{V}_C\cos\varphi\,\hat{e}_t + V_C\cdot\cos\varphi\,\dot{\theta}\,\hat{e}_n - \dot{V}_C\sin\varphi\,\hat{e}_n - V_C\cdot\sin\varphi\left(-\dot{\theta}\,\hat{e}_t\right)\right] \Rightarrow$$

$$\boxed{a_G = \left(\dot{\theta}^2 d_F + \dot{V}_C\cos\varphi + V_C\cdot\sin\varphi\,\dot{\theta}\right)\hat{e}_t + \left(-\ddot{\theta}d_F + V_C\cos\varphi\,\dot{\theta} - \dot{V}_C\sin\varphi\right)\hat{e}_n.}$$

Back to AMB: $(*) \Longrightarrow$

$$-dR_N\hat{k} = -d_F\hat{e}_t \times m\vec{a}_G + I^G\ddot{\theta}\hat{k} \xRightarrow{\hat{e}_t\times\hat{e}_n \text{ :survives}}$$

$$-dR_N\hat{k} = m\ddot{\theta}d_F^2\hat{k} + mV_c\cos\varphi\dot{\theta}(-d_F)\hat{k} + md_F\dot{V}_c\sin\varphi\hat{k} + I^G\ddot{\theta}\hat{k} \Rightarrow$$

$$\ddot{\theta}(I^G + d_F^2 m) = md_F V_c\dot{\theta}\cos\varphi - m\dot{V}_c d_F\sin\varphi - dR_N \Rightarrow$$

$$\boxed{\ddot{\theta} = \frac{md_F(+V_c\dot{\theta}\cos\varphi - \dot{V}_c\sin\varphi) - dR_N}{I^G + d_F^2 m}} \longrightarrow \text{If } \varphi = 0 \text{ and } R_N = 0:$$

$$\ddot{\theta} = \frac{+d_F mV_c\dot{\theta}}{I^G + md_F^2}.$$

Stuff we don't know yet: $\dot{V}_c$, R_N

This d.e. is of the form $\dot{y} = c\cdot y$, $c>0$ (for $V_c>0$). i.e., it is UNSTABLE, as opposed to the one in the lecture. The reason is that now the COM is not in front of the rolling wheel, but behind it. It is stable for $V_c<0$, as will be demonstrated in simulations.

O.K.

LMB: $\sum\vec{F} = m\cdot\vec{a}_G \Rightarrow \{R_N\hat{e}_n + R_T\hat{e}_t + F_N\cos\varphi\,\hat{e}_n + F_n\sin\varphi\,\hat{e}_t\} = m\vec{a}_G$

$$\cancel{\{\dots\}\cdot\hat{e}_n \Rightarrow R_N + F_N\cos\varphi = m(-\ddot{\theta}d_F + V_c\cos\varphi\dot{\theta} - \dot{V}_c\sin\varphi)}$$ wrong unit vector...

$$\{\dots\}\cdot\frac{\vec{V}_c}{\|\vec{V}_c\|} \Rightarrow R_N(-\sin\varphi)+R_T\cos\varphi+\cancel{F_N\cos\varphi(-\sin\varphi)}+$$

$$+\cancel{F_N\sin\varphi\cos\varphi} = m\vec{a}_G\frac{\vec{V}_c}{\|\vec{V}_c\|} = m(\dot\theta^2 d_F\cos\varphi+\dot V_c\cos^2\varphi+\cancel{V_c\sin\varphi\cos\varphi\dot\theta}$$

$$+\ddot\theta d_F\sin\varphi-\cancel{V_c\cos\varphi\sin\varphi\dot\theta}+\dot V_c\sin^2\varphi) \Rightarrow$$

$$\boxed{\dot V_c = \frac{1}{m}\left[-R_N\cdot\sin\varphi+R_T\cos\varphi\right]-\left(\dot\theta^2 d_F\cos\varphi+\ddot\theta d_F\sin\varphi\right)}$$

↳ If no friction ($R_N=R_T=0$) and $\varphi=0$:

$$\dot V_c = -\dot\theta^2 d_F \text{ (like in lecture, but opposite sign).}$$

normal to $\vec{V}_c$

$$\{\dots\}\cdot\frac{\vec{F}_N}{\|\vec{F}_N\|} \Rightarrow R_N\cos\varphi+R_T\sin\varphi+F_N\cos\varphi\cos\varphi+F_N\sin\varphi\sin\varphi =$$

$$= m(\dot\theta^2 d_F\sin\varphi+\cancel{\dot V_c\cos\varphi\sin\varphi}+V_c\cdot\sin\varphi\sin\varphi\dot\theta +$$

$$-\ddot\theta d_F\cos\varphi+V_c\cos\varphi\cos\varphi\dot\theta-\cancel{\dot V_c\sin\varphi\cos\varphi} \Rightarrow$$

$$\boxed{F_N = -R_T\sin\varphi-R_N\cos\varphi+m\left[V_c\dot\theta+\dot\theta^2 d_F\sin\varphi-\ddot\theta d_F\cos\varphi\right]}$$

↳ For $R=0$ (no friction) and $\varphi=0$:

$$F_N = m\left[V_c\dot\theta-d_F\ddot\theta\right] \text{ (like in lecture, but opposite sign for } \ddot\theta\text{)}$$

Now, let's substitute $\varphi = 0$ ($\cos\varphi = 1$, $\sin\varphi = 0$)

$$\boxed{\begin{aligned} \ddot{\theta} &= + \frac{m d_F V_c \dot{\theta} - d R_N}{I^G + d_F^2 m} \\ \dot{V}_c &= - d_F \dot{\theta}^2 + \frac{1}{m} R_T \end{aligned}} \quad (\star) \quad , \quad F_N = m\left[V_c \dot{\theta} - d_F \ddot{\theta}\right] - R_N$$

<u>Friction:</u> $\vec{R} = -\mu N \frac{\vec{V}_R}{\|\vec{V}_R\|}$, $\vec{V}_R \longrightarrow$ velocity of rear wheel (R)

where $\mu \in (0.1, 1)$ and $\boxed{N = N_R = \frac{d_F}{d_F + d_R} \cdot mg}$, and

$$\vec{V}_R = V_c \cdot \hat{e}_t + (\dot{\theta} k) \times \vec{r}_{R/C} = V_c \cdot \hat{e}_t + (\dot{\theta} \hat{k}) \times (-d\, \hat{e}_t) \Rightarrow$$

$$\vec{V}_R = V_c \cdot \hat{e}_t + (-d\dot{\theta})\, \hat{e}_n$$

Thus:

$$R_T = \vec{R} \cdot \hat{e}_t = -\mu N \frac{\vec{V}_R}{\|\vec{V}_R\|} \cdot \hat{e}_t \Rightarrow \boxed{R_T = -\mu N \cdot \frac{V_c}{\|\vec{V}_R\|}}$$

$$R_N = \vec{R} \cdot \vec{e}_n = -\mu N \frac{\vec{V}_R}{\|\vec{V}_R\|} \cdot \vec{e}_n \Rightarrow \boxed{R_N = -\mu N \frac{(-d\dot{\theta})}{\|\vec{V}_R\|}}$$

(D) Setup numerical simulation:

$$\left\{ \begin{aligned} \dot{x}_c &= V_c \cdot \cos\theta \\ \dot{y}_c &= V_c \cdot \sin\theta \\ \dot{\theta} &= \omega \\ \dot{V}_c &= \cdots \star \\ \ddot{\theta} = \dot{\omega} &= \cdots \star \end{aligned} \right\}$$

Parameters used in simulations:

$m = 1$, $I^G = 1$, $g = 1$, $\mu \in (0.1, 1)$, $d = 1$, $d_F = 0.6$, $d_R = 0.4$,

Initial conditions:

$q_0 = [0 \quad 0 \quad 30°]^T$

$_0 = [+1 \quad +0.1]^T \rightarrow$ plus others

Coden and figures attached $\longrightarrow$

Contents

- Initialization
- ODE Solver
- Plots
- Animation


```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% MAE 5735: Intermediate Dynamics & Vibrations
%
%
% Problem Set #5: Problem 16 - Braking stability (Car with locked brakes)
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% This simulation is for locked REAR brakes and fixed front steering!
```

Initialization

```
close all; clear all; clc

global p                                % Instead of passing 'p' to function

p.m  = 1;                               % mass
p.Ig = 1;                               % inertia
p.g  = 1;                               % gravity
p.mu = 0.5;                             % friction coefficient
p.d  = 1;                               % wheelbase d = df + dr
p.df = 0.6*p.d;                         % distance of G from front wheels
p.dr = 0.4*p.d;                         %   -//-      -//-   rear wheels

tfinal   = 10;
n        = 2;
timestep = 10^(-n);                     % NOT the timestep of ode23()
timespan = 0:timestep:tfinal;

q0     = [0 0 pi/6]';                   % (x, y, theta)
u0     = [+1 +0.1]';                    % (V, omega)
z0     = [q0; u0];                      % initial state vector

disp('My "Car with REAR brakes blocked" code is now running!');
disp(' ')
```

ODE Solver

```
tStart              = tic;
% I only want the "stop" event if there is friction, otherwise continue:
if p.mu > 1e-5;
    options         = odeset('reltol', 1e-8, 'abstol', 1e-8, 'events', @event_stop);
```

```
enlarge = pixelsize*img_size;
axis([min(x)-enlarge(1) max(x)+enlarge(1) min(y)-enlarge(2) max(y)+enlarge(2)])

% pause(1)
for t = 1:5:length(time)

    car_now  = 255 - imrotate(255-car_img, -180/pi*(theta(t)));
    curA     = imrotate(imA, -180/pi*(theta(t)));                          % Rotat
    img_size = size(car_now);

    nx  = (0:pixelsize:(pixelsize*(img_size(2)-1)))-(img_size(2)-1)*pixelsize/2;
    ny  = (0:pixelsize:(pixelsize*(img_size(1)-1)))-(img_size(1)-1)*pixelsize/2;
    car = image('CData',car_now,'XData',nx+x(t),'YData',ny+y(t), 'AlphaData', curA);    % Displ

    pause(eps)                                                             % Pause
    delete(car)                                                            % Delet
end
car = image('CData',car_now,'XData',nx + x(t),'YData',ny + y(t), 'AlphaData', curA);
hold off

disp('Execution terminated.'); disp(' ')

% END
```


Note: The car is NOT plotted ~~exactly~~ at the CoM ~~exact~~ but at C. ✓

```
function zdot = car_rear_brakes(t, z)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
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%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% This simulation is for locked REAR brakes and fixed front steering!

global p

% x = z(1);
% y = z(2);
theta = z(3);
V     = z(4);
omega = z(5);

N  = p.m*p.g*p.df/(p.d);
VR = [V -p.d*omega]';
% RT = p.mu*N*sign(V);
% RN = p.mu*N*sign(-omega);
R  = -p.mu*N*VR/norm(VR);
RT = R(1); RN = R(2);

xdot   = V*cos(theta);
ydot   = V*sin(theta);
th_dot = omega;
Vdot   = -p.df*omega^2 +(1/p.m)*RT;
Wdot   = (1/(p.Ig + p.df^2*p.m))*(p.m*p.df*V*omega - p.d*RN);

zdot   = [xdot ydot th_dot Vdot Wdot]';

end

function [value, done, dir] = event_stop(t, z)

% Stop simulation when Vc <= 10^(-3)
value = abs(z(4)) -1e-2; done = 1; dir = -1;

end
```
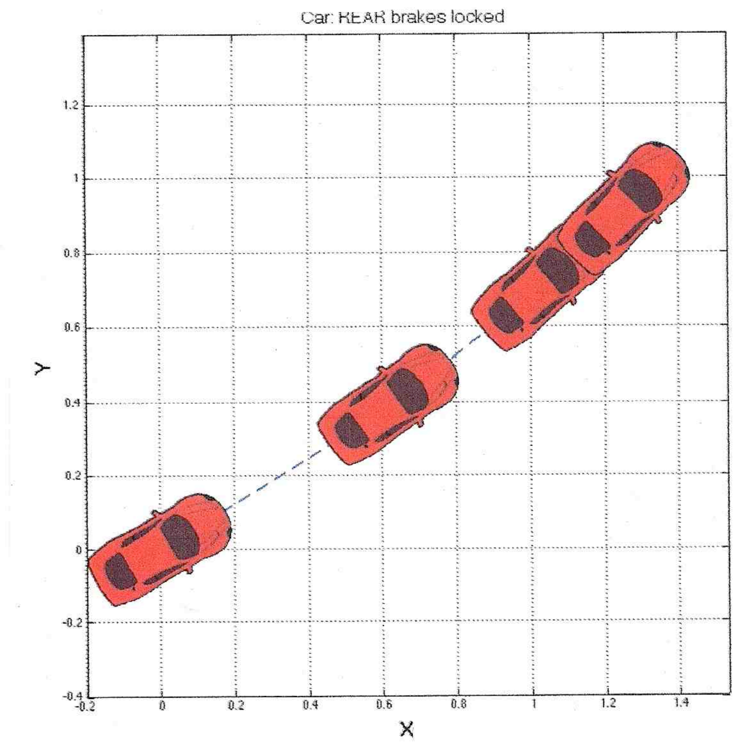
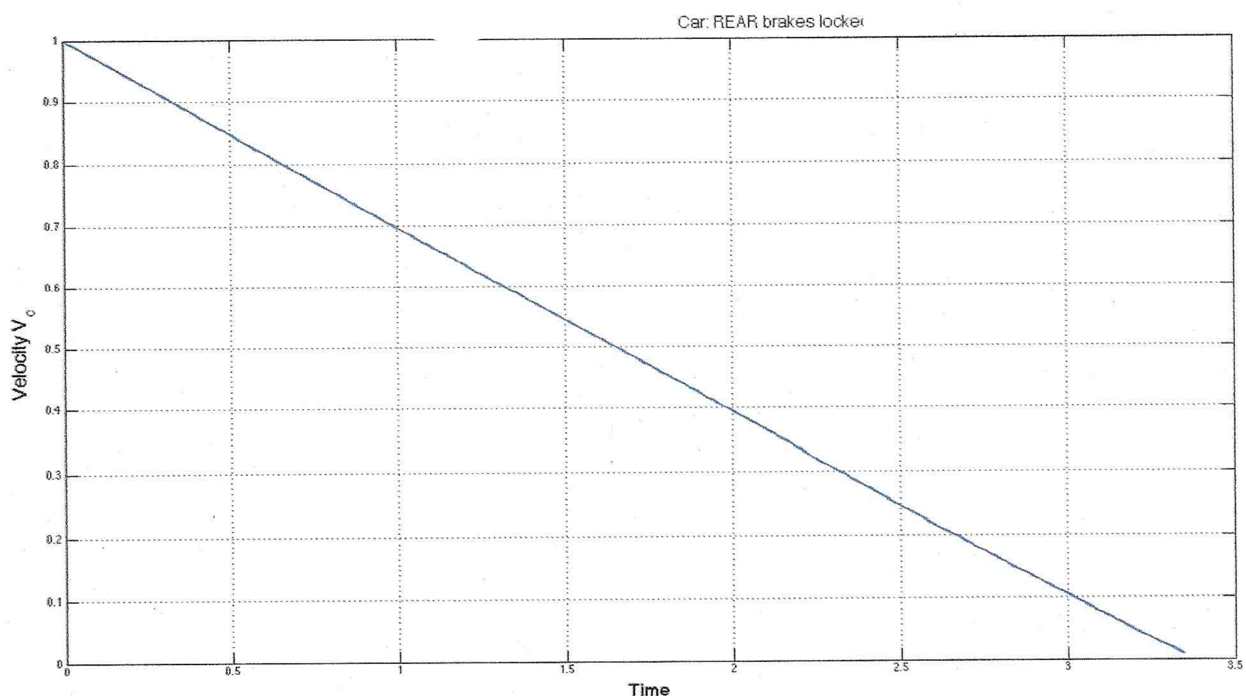
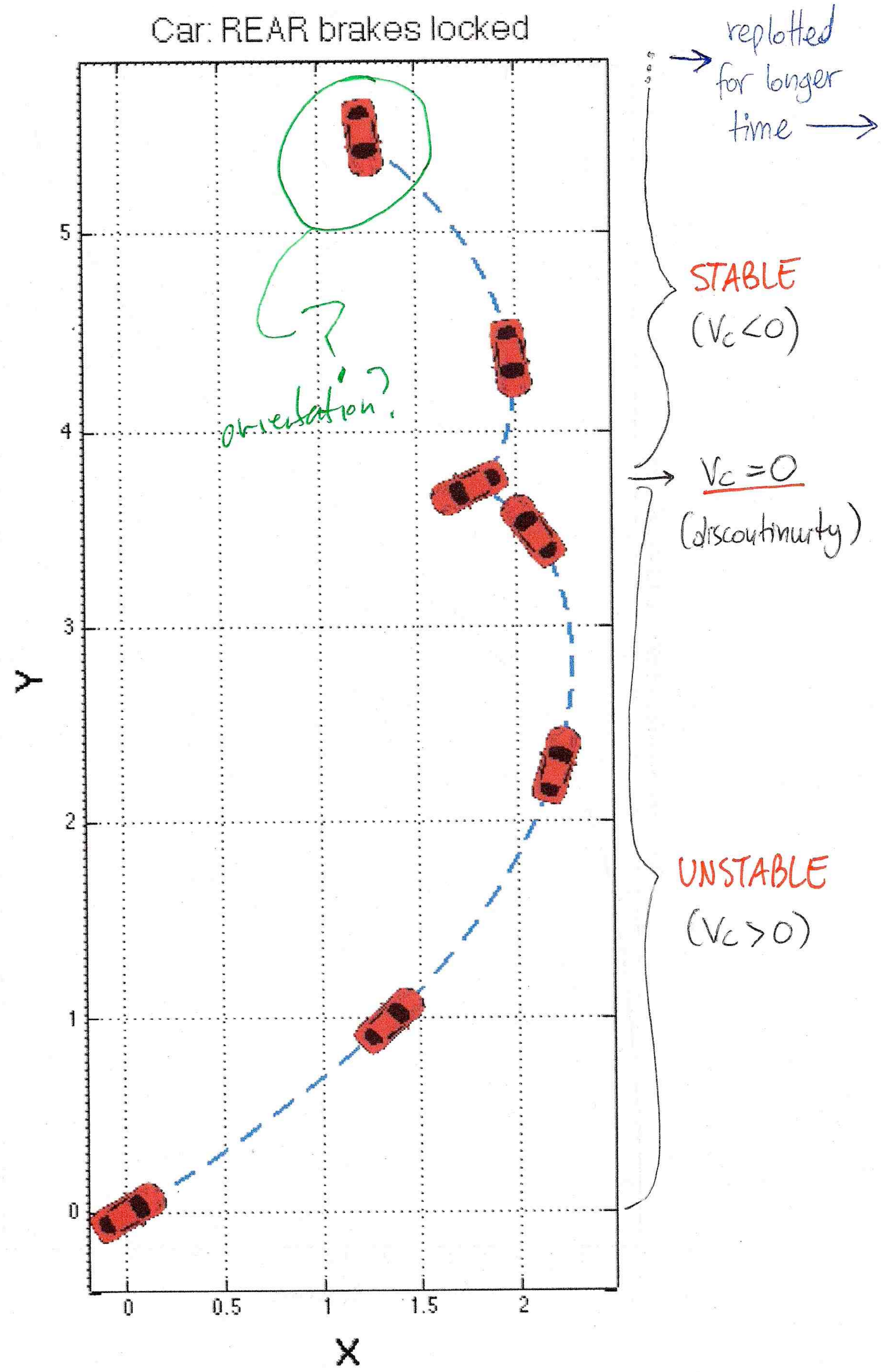

(C) Plots



$V_c(0) = +1$, $\omega(0) = +0.1$, $\mu = 0.5$

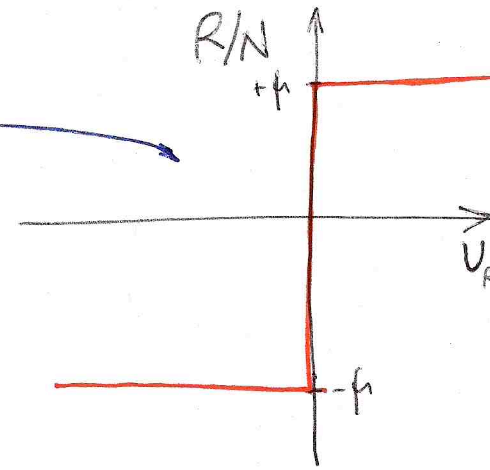
(decellerates will turing -unstable- and stops when $V_c = 0$)

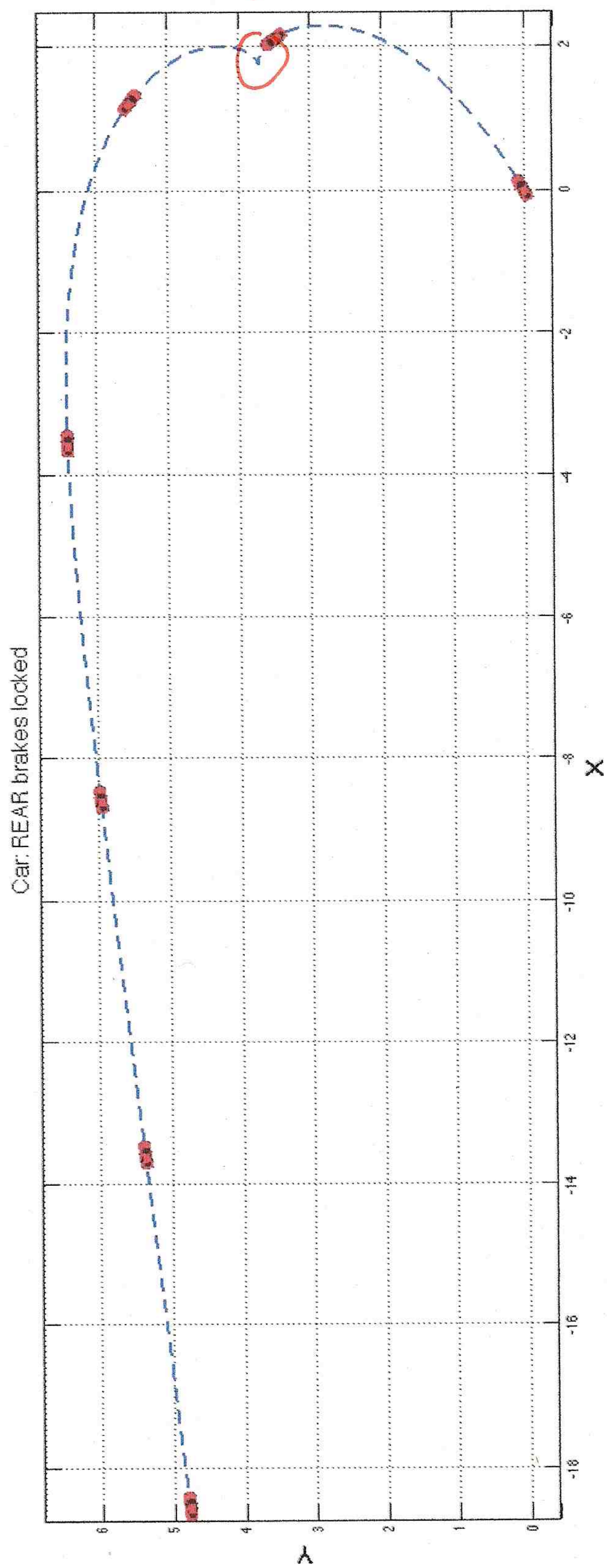




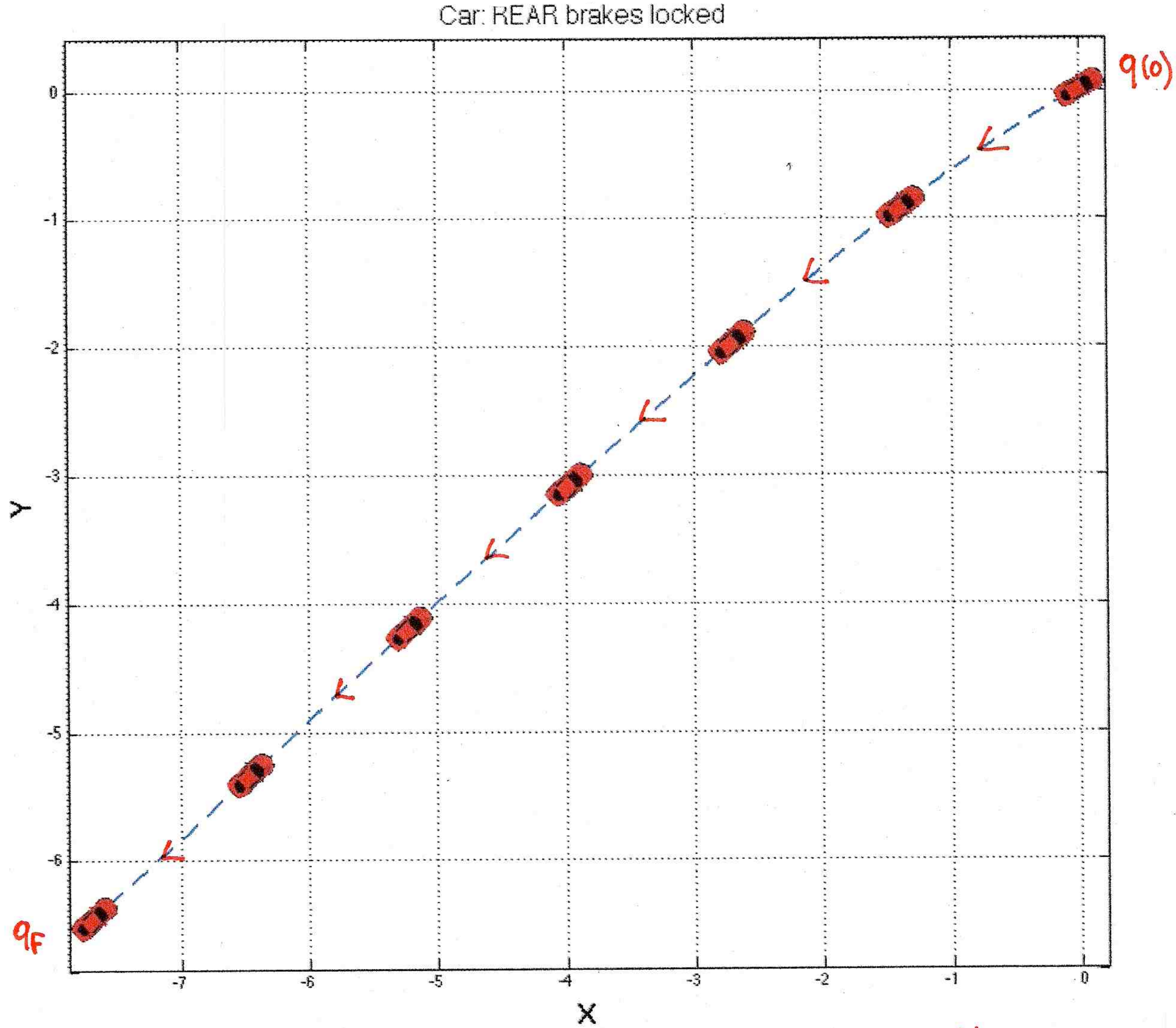
$V_c(0) = +1$, $\omega(0) = +0.1$, $\boxed{\mu = 0}$ No friction!

Flip (discontinuity ...)

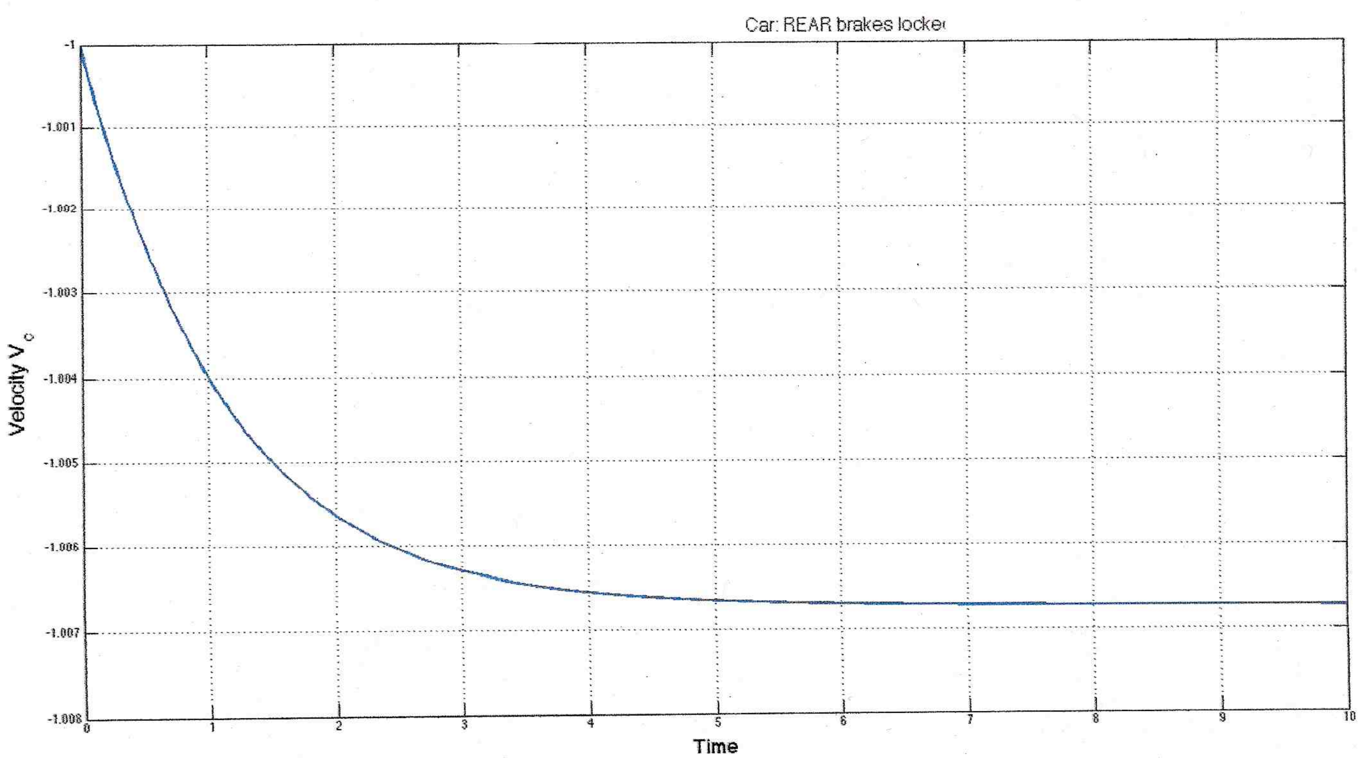




Same as p.14a but longer time to demonstrate stable behavior. after flip.



$V_c(0) = -1$, $\omega(0) = +0.1$, $\boxed{\mu = 0}$ No friction STABLE



(d) Analytical results wrt steering stability

Let's try the kinetic energy of the car as a Lyapunov function candidate:

$$E_K = \tfrac{1}{2} m V_c^2 + \tfrac{1}{2}(I^G + m d_F^2)\omega^2$$

* E_K is positive definite, i.e., $E_K(V_c=0, \omega=0)=0$ and $E_K \geq 0$ for $V_c, \omega \neq 0$ (globally)

* E_K is radially unbounded, i.e., $\left\| \begin{matrix} V_c \\ \omega \end{matrix} \right\| \to \infty \Rightarrow E_K \to \infty$

* Assuming E_K is smooth (which it is NOT due to Coulomb friction):

$$\dot{E}_K = \frac{d}{dt} E_K = m V_c \dot{V}_c + (I^G + m d_F^2)\omega\dot{\omega} \xrightarrow{\dot{V}_c, \ddot{\theta}}$$

$$\dot{E}_K = m V_c\left(-d_F \omega^2 + \tfrac{1}{m} R_T\right) + \cancel{(I^G + m d_F^2)}\,\omega \cdot \frac{m d_F V_c \omega - d R_N}{\cancel{I + m d_F^2}} =$$

$$= -\cancel{m d_F \omega^2 V_c} + V_c R_T + \cancel{m d_F \omega^2 V_c} - d\omega R_N = V_c R_T - d\omega R_N$$

$$\xRightarrow{R_T, R_N} \dot{E}_K = V_c\left(-\mu N \frac{V_c}{\|\vec{V}_R\|}\right) - d\omega\left(-\mu N \frac{(-d\omega)}{\|\vec{V}_R\|}\right) =$$

$$= -\mu N \frac{1}{\|\vec{V}_R\|} \cdot (V_c^2 + d^2\omega^2) = -\mu N \frac{1}{\|\vec{V}_R\|} \|\vec{V}_R\|^2 \Rightarrow$$

$$\dot{E}_K = -\mu N \|\vec{V}_R\| \Rightarrow \boxed{\dot{E}_K = -\mu N \cdot \sqrt{V_c^2 + d^2\omega^2}}$$

$\Rightarrow$ Energy is conserved

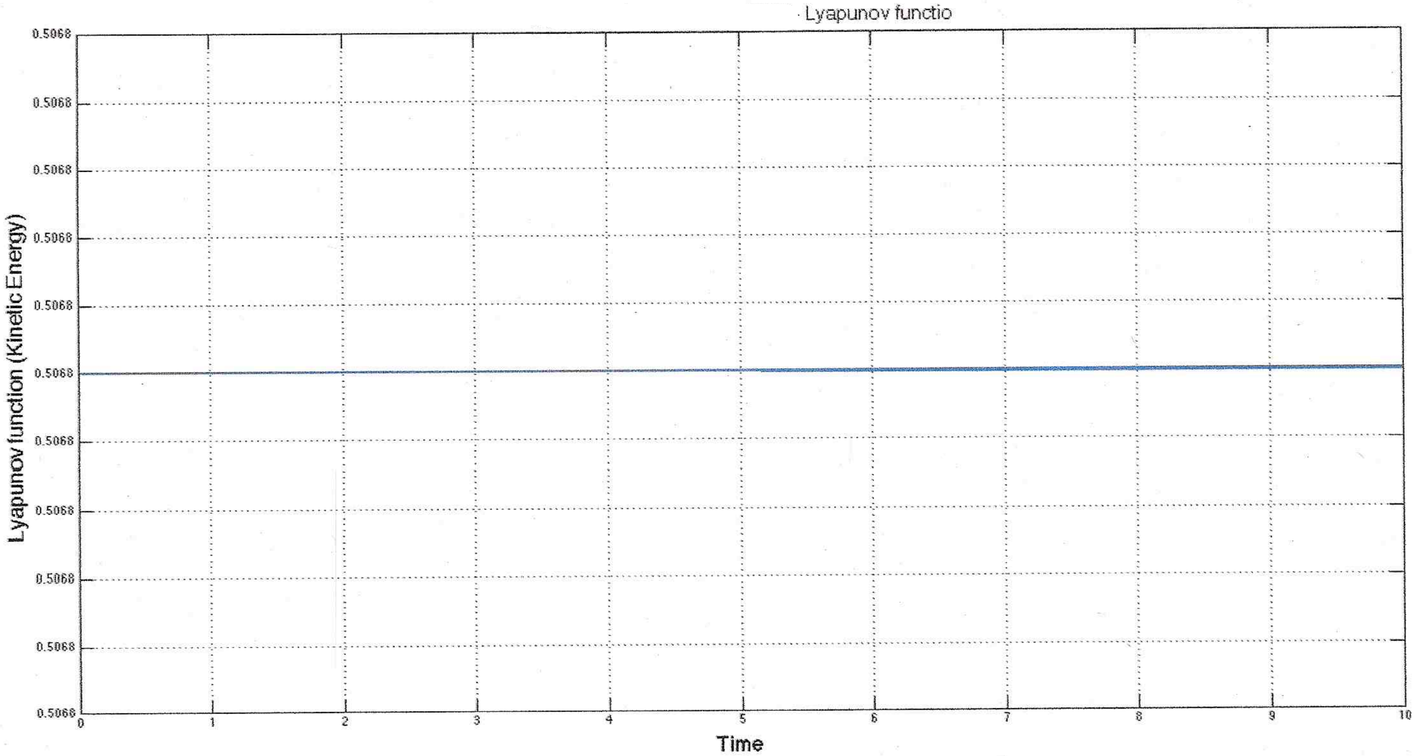
* For $\mu > 0$, $\boxed{\dot{E}_K < 0}$ $\Rightarrow$ Lyapunov function strictly decreasing.

Therefore the car converges to $\boxed{(V_c = 0, \omega = 0)}$. Since $\dot{E}_K(V_c, \omega) < 0$, $\forall (V_c, \omega) \in \mathbb{R}^2 \setminus \{0\}$, the car is Globally Asymptotically Stable.

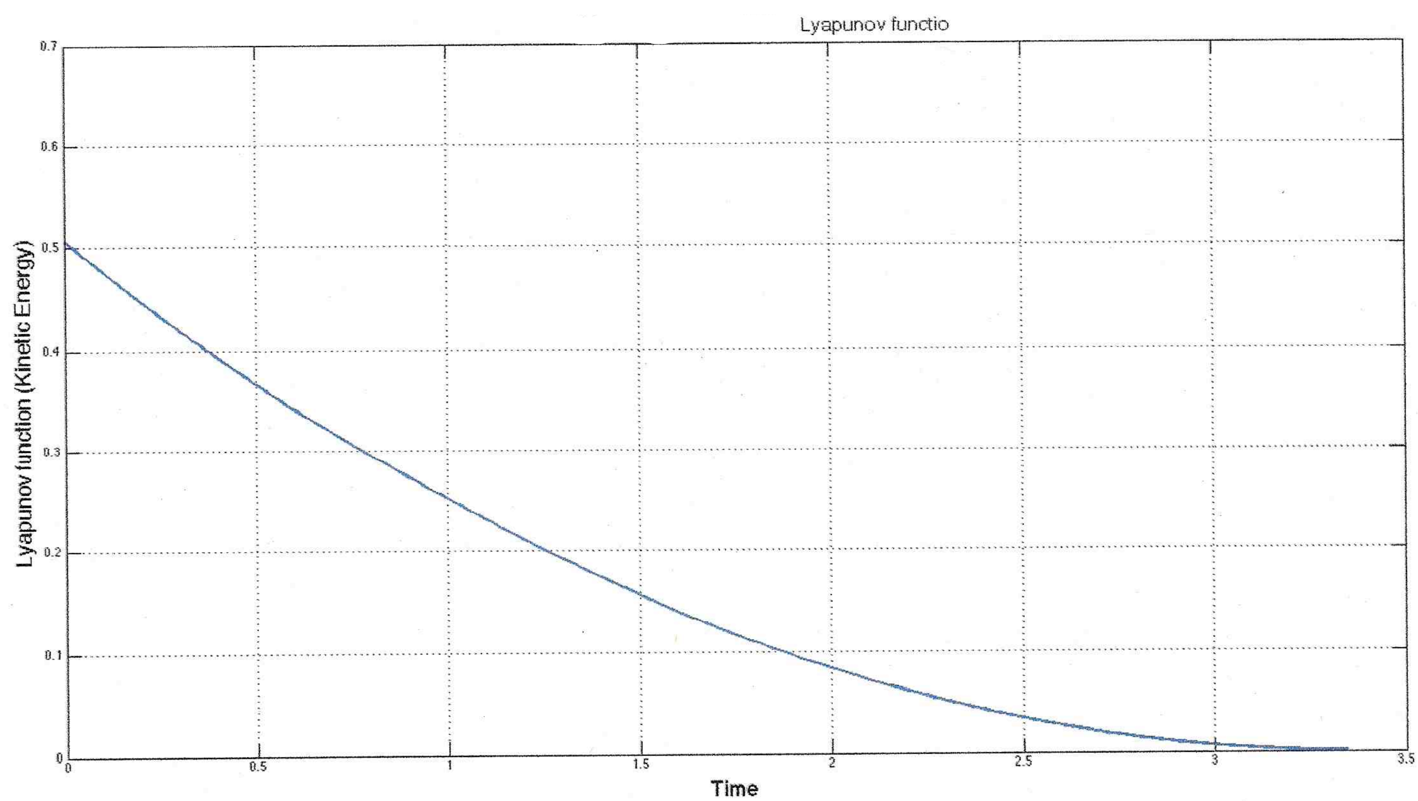
This result doesn't say much about what happens from t=0 to the time the car stops $(V_c = 0, \omega = 0)$, but it says that from any x_0, y_0, θ_0 (due to symmetry) and any $V_c(0), \omega(0)$, it will converge to $V_c = \omega = 0$.

Note: We have assumed system is smooth. In reality there is a discontinuous right-hand side due to Coulomb friction. (See Shevitz, Paden, "Lyapunov Stability Theory of Nonsmooth Systems", IEEE Tr. on Automatic Control, Vol. 39, No. 9, 1994)

Two illustrative plots are presented next $\rightarrow$



$\mu = 0 \quad , \dot{E}_k = 0$



$\mu = 0.5 \quad , \dot{E}_k < 0$