1) 2D. A uniform disk (mass M, radius r) rolls without slip on level ground. Handing from it's center C is a pendulum (with mass m, moment of inertia I about it's COM, and distance d from C to the pendulum's COM at G). Answer all questions in terms of $x, \dot{x}, \theta, \dot{\theta}, \hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, M, m, r, d$ and I, or an appropriate simplified subset of these.

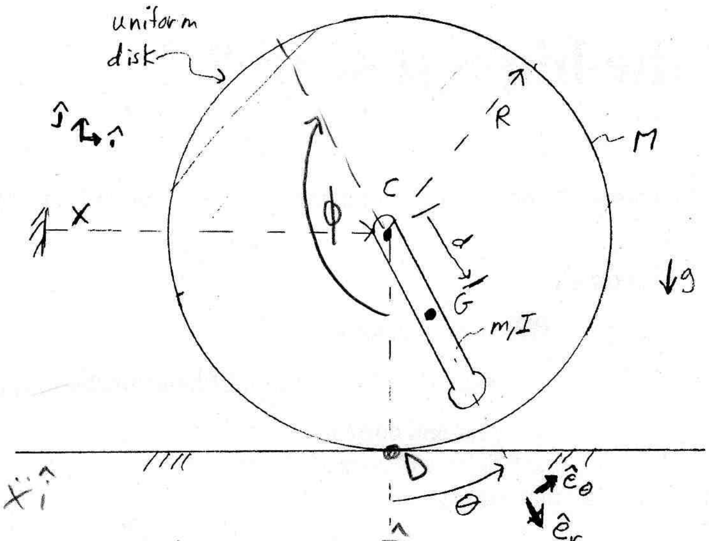
(1) Write 2 scalar equations from which one could solve for $\ddot{x}$ and $\ddot{\theta}$.

(2) For small motions near $x = 0$ and $\theta = 0$ write the equations in standard vibration form, finding the components of the matrices M and K.

(3) By *any* means find just one normal mode: give the value of ω, the components of $\vec{v}$, and describe the mode in words.

Notation: $\vec{Z}_{P/K} = \vec{Z}_P$ where K is the inertial frame and Z is any vector.

Instead of $\hat{e}_r \Rightarrow \hat{\lambda}$
$\hat{e}_\theta \Rightarrow \hat{n}$



$x = \phi R$

and $\vec{\omega}_1 = -\dot{\phi}\hat{k}$

Disc angular velocity

$\vec{r}_c = x\hat{\imath}$ $\quad \vec{V}_c = \dot{x}\hat{\imath}$ $\quad \vec{a}_c = \ddot{x}\hat{\imath}$

$\vec{r}_{G/C} = d\hat{\lambda}$ $\quad \vec{V}_{G/C} = d\dot{\theta}\hat{n}$ $\quad \vec{a}_{G/C} = d\ddot{\theta}\hat{n} - d\dot{\theta}^2\hat{\lambda}$

Consider Rod as a system

AMB/C

$\sum M_C = d\hat{\lambda} \times -mg\hat{\jmath}$

$= -mgd\sin\theta\hat{k}$

$\dot{\vec{H}}_{/C} = [\vec{r}_{G/C} \times m\vec{a}_G] + I\ddot{\theta}\hat{k}$

$= \vec{r}_{G/C} \times m[\ddot{x}\hat{\imath} + d\ddot{\theta}\hat{n} - d\dot{\theta}^2\hat{\lambda}] + I\ddot{\theta}\hat{k}$

$= md\cos(\theta)\ddot{x}\hat{k} + (md^2+I)\ddot{\theta}\hat{k}$

$\sum M_{/C} = \sum \dot{H}_{/C}$

$-mgd\sin\theta\hat{k} \quad md\cos\theta\ddot{x}\hat{k} + (md^2+I)\ddot{\theta}\hat{k}$

$\boxed{md\cos(\theta)\ddot{x} + (I+md^2)\ddot{\theta} + mgd\sin\theta}$

Consider rod and cylinder as system

AMB/D $\quad \vec{r}_{C/D} = R\hat{\jmath}$ $\quad \vec{r}_{G/D} = d\hat{\lambda} + R\hat{\jmath}$

$\sum M_D = \vec{r}_{G/D} \times (-mg)\hat{\jmath}$

$= (d\hat{\lambda} + R\hat{\jmath}) \times (-mg\hat{\jmath})$

$= -mgd\sin\theta\hat{k}$

$$\vec{H}_{/D} = \vec{r}_{C/D} \times M\vec{a}_C + \frac{MR^2}{2}(-\ddot{\phi})\hat{k} + \vec{r}_{G/D} \times m\vec{a}_G + I\ddot{\theta}\hat{k}$$ ②

$$= R\hat{j} \times M\ddot{x}\hat{i} - \frac{MR^2}{2}\ddot{\phi}\hat{k} + (d\hat{\lambda} + R\hat{j}) \times m[\ddot{x}\hat{i} + d\ddot{\theta}\hat{n} - d\dot{\theta}^2\hat{\lambda}] + I\ddot{\theta}\hat{k}$$

$$= -MR\ddot{x}\hat{k} - \frac{MR^2}{2}\ddot{\phi}\hat{k} + md[\cos\theta\,\ddot{x} + d\ddot{\theta}]\hat{k} + mR[-\ddot{x} - \cos\theta\, d\ddot{\theta} + d\dot{\theta}^2\sin\theta]\hat{k} + I\ddot{\theta}\hat{k}$$

$$= \Big[-MR\ddot{x} - \frac{MR}{2}\ddot{x} + md\cos\theta\,\ddot{x} - mR\ddot{x} + md^2\ddot{\theta} - mRd\cos\theta\,\ddot{\theta} + I\ddot{\theta} + mRd\dot{\theta}^2\sin\theta\Big]\hat{k}$$

$$= \Big[\Big[-\frac{3}{2}MR + md\cos\theta - mR\Big]\ddot{x} + \Big[I + md^2 - mRd\cos\theta\Big]\ddot{\theta} + mRd\dot{\theta}^2\sin\theta\Big]\hat{k}$$

$$\therefore \Sigma\vec{M}_{/D} = \dot{\vec{H}}_{/D} \Rightarrow \Big[-\frac{3}{2}MR + md\cos\theta - mR\Big]\ddot{x} + \Big[I + md^2 - mRd\cos\theta\Big]\ddot{\theta} + md\sin\theta[g + \dot{\theta}^2R] = 0 \quad -②$$

$\therefore$ The equations are

$$md\cos\theta\,\ddot{x} + (I + md^2)\ddot{\theta} + mgd\sin\theta = 0 \quad -①$$

$$\Big[-\frac{3}{2}MR + md\cos\theta - mR\Big]\ddot{x} + \Big[I + md^2 - mRd\cos\theta\Big]\ddot{\theta} + md\sin\theta[g + \dot{\theta}^2R] = 0 \quad ②$$

This can be simplified by ① − ② ✓+

$$\Rightarrow \Big[\frac{3}{2}M + m\Big]R\ddot{x} + mRd\cos\theta\,\ddot{\theta} - mdR\dot{\theta}^2\sin\theta = 0$$

$$\Rightarrow \Big[\frac{3}{2}M + m\Big]\ddot{x} + md\cos\theta\,\ddot{\theta} - md\dot{\theta}^2\sin\theta = 0$$

Thus new set of equations is

$$\boxed{\begin{aligned} &md\cos\theta\,\ddot{x} + (I + md^2)\ddot{\theta} + mgd\sin\theta = 0 \quad ① \\ &\Big[\frac{3}{2}M + m\Big]\ddot{x} + md\cos\theta\,\ddot{\theta} - md\dot{\theta}^2\sin\theta = 0 \quad ② \end{aligned}}$$ answer to part (a)

b) for small oscillations $\theta \approx 0$ and $\dot{\theta}^2 \approx 0$ $\sin\theta \approx 0$, $\cos\theta \approx 1$

$$\Rightarrow md\ddot{x} + (I + md^2)\ddot{\theta} + mgd\theta = 0$$

$$\Big[\frac{3M}{2} + m\Big]\ddot{x} + md\ddot{\theta} = 0$$

$$\Rightarrow \underbrace{\begin{bmatrix} I + md^2 & md \\ md & \frac{3M}{2} + m \end{bmatrix}}_{M} \begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} + \underbrace{\begin{bmatrix} mgd & 0 \\ 0 & 0 \end{bmatrix}}_{K} \begin{bmatrix} \theta \\ x \end{bmatrix} = 0$$ ✓

c)

$$M^{-1} = \frac{1}{\det}\begin{bmatrix} \frac{3M}{2} + m & -md \\ -md & I + md^2 \end{bmatrix} \qquad \det = \frac{3IM}{2} + \frac{3Mmd^2}{2} + Im$$

$$M^{-1}K = \frac{1}{\det}\begin{bmatrix} (\frac{3M}{2} + m)mdg & 0 \\ -m^2gd^2 & 0 \end{bmatrix}$$

Goto back side for complete answer.

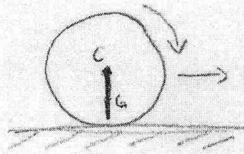
Clearly this system as two eigen values out of which one is zero

$$\omega_1^2 = 0$$

$$\omega_2^2 = \frac{\left(\frac{3}{2}M + m\right) m d g}{I\left(\frac{3M}{2} + m\right) + \frac{3M m d^2}{2}}$$

& $V_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

This motion represents when θ is not changing at all times and disk is rolling



This the motion corresponding to $\omega = 0$

also for ω_2^2 when $M \to \infty$

Good check

$$\lim_{M\to\infty} \omega_2^2 = \frac{\frac{3}{2} m d g}{\frac{3}{2} I + \frac{3}{2} m d^2} = \frac{m d g}{I + m d^2}$$

Which gives the frequency of pysical pendulum.

Governing equations using Lagrange equations. ok,

$\vec{V}_G = \dot{x}\hat{i} + d\dot{\theta}\hat{n} = (\dot{x} + d\dot{\theta}\cos\theta)\hat{i} + d\dot{\theta}\sin\theta\,\hat{j}$

$|\vec{V}_G|^2 = \dot{x}^2 + d^2\dot{\theta}^2 + 2\dot{x}d\dot{\theta}\cos\theta$

$\vec{V}_c = \dot{x}\hat{i} \Rightarrow |\vec{V}_c|^2 = \dot{x}^2$

Kinetic energy of rod $= \frac{1}{2}m|V_G|^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}m(\dot{x}^2 + d^2\dot{\theta}^2 + 2\dot{x}d\dot{\theta}\cos\theta) + \frac{1}{2}I\dot{\theta}^2$

$= \frac{1}{2}(md^2 + I)\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2 + m\dot{x}\dot{\theta}d\cos\theta$

Kinetic energy of ~~cylinder~~ disc $= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}\frac{MR^2}{2}\dot{\phi}^2 = \frac{1}{2}M\dot{x}^2 + \frac{M}{4}\dot{x}^2$

$= \frac{3}{4}M\dot{x}^2$

$\therefore T = \frac{1}{2}\left(\frac{3}{2}M + m\right)\dot{x}^2 + \frac{1}{2}(I + md^2)\dot{\theta}^2 + m\dot{x}\dot{\theta}d\cos\theta$

(4) $V = -mgd\cos\theta$

$\mathcal{L} = T - V$

$$\mathcal{L} = \frac{1}{2}\left[\frac{3M}{2} + m\right]\dot{x}^2 + \frac{(I + md^2)}{2}\dot{\theta}^2 + md\dot{\theta}\dot{x}\cos\theta + mdg\cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$$

$$\Rightarrow \left(\frac{3}{2}M + m\right)\ddot{x} + md\ddot{\theta}\cos\theta - md\dot{\theta}^2\sin\theta = 0 \quad \text{—(1)}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0$$

$$\Rightarrow (I + md^2)\ddot{\theta} + md\cos\theta\,\ddot{x} + mgd\sin\theta = 0 \quad \text{—(2)}$$

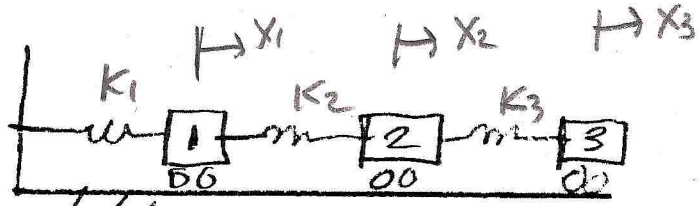
This is exactly the same system of equation which we have obtained using AMB.

2) Consider the one-D arrangement of three unequal masses and three unequal springs shown. Write Matlab code that would

plot the deflection (from equilibrium) of the first mass as a function of time.

Pick non-trivial numerical values for all variables (that is, do not pick variables that especially simplify the problem).

- The initial deflections of the masses are given as $\vec{x}_0 = [111]'$.
- The initial velocities are zero.
- Use techniques from vibrations (i.e., not ode23 or Euler's method).
- As much as possible, have Matlab do the calculations (i.e., don't try to find normal modes by hand calculations or intuition).
- You can assume that none of the normal modes have $\omega = 0$.



FBDS

$$k_1 x_1 \leftarrow \square \rightarrow k_2(x_2 - x_1)$$

$$k_2(x_2-x_1) \leftarrow \square \rightarrow k_3(x_3-x_2)$$

$$k_3(x_3-x_2) \leftarrow \square$$

$$\underbrace{\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}}_{M} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \underbrace{\begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}}_{K} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

MATLAB code

```
m1 = 7, m2 = 13, m3 = 39;
k1 = 2, k2 = 3, k3 = pi;
M = diag([m1 m2 m3]);
```

2) Cont'd

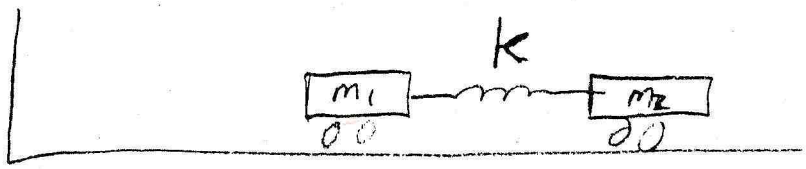
Note: initial vel. = [0 0 0] ⇒ we can use only cosine waves

```
t = linspace (0, 50, 500); % time for plotting
K = [ (K1 +K2)   -K2        0
      -K2        (K2 +K3)   -K3
      0          -K3        K3 ];

[V D] = eig (K, M);
 ↑    ↑ e-values
 modes

X0 = [1 1 1]'; % init. conds
r = V \ X0 % find modal amplitudes
           % because at t=0  X0 = V·r
w = sqrt (diag (D)); % angular frequencies

X1 = V(1,1) * r1 * cos (w1 t) ...
   + V(1,2) * r2 * cos (w2 t) ...
   + V(1,3) * r3 * cos (w3 t);

plot (t, X1)
```

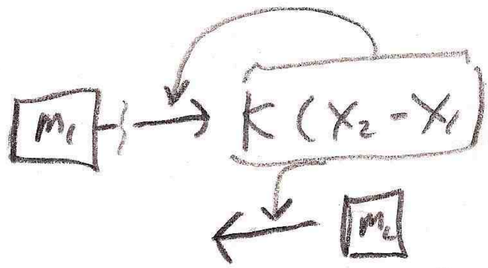
% Note: if we used X0 = V(:, 1) plot would be a sine wave (I.C. would be a normal mode)

or 2 or 3

3) 1D. Two *unequal* masses are connected to each other, and nothing else, by one linear spring. Find and describe as many normal modes as you can. That is, clearly give the mode shapes and frequencies in terms of m_1, m_2 and k. As always, clearly justify your results.



FBDs:



ONE mode: $\underbrace{\text{steady}}_{\text{rigid}}$, translation $\Rightarrow \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\omega = 0$

For the other take C.O.M. to be stationary

$\Rightarrow \quad m_1 V_1 = -m_2 V_2 \Rightarrow V_2 = -\frac{m_1}{m_2} V_1$

say $\boxed{\vec{V} = \begin{bmatrix} 1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -m_1/m_2 \end{bmatrix}}$

$$\left(\frac{K}{m}\right)^{eff} = \frac{F/\delta}{m}$$

use either mass, say 1:

$$= \frac{K(1 + m_1/m_2)/1}{m_1}$$

$$= K\left(\frac{1}{m_1} + \frac{m_1}{m_1 m_2}\right)$$

$$= K\,(m_1 + m_2)/m_1 m_2$$

$$= \frac{k}{\underbrace{\left\{\left(\frac{m_1 m_2}{m_1 + m_2}\right)\right\}}}$$

also called "effective mass" but it's a diff. thing.

$$\Rightarrow \boxed{\omega = \sqrt{\frac{K(m_1 + m_2)}{m_1 m_2}}}$$