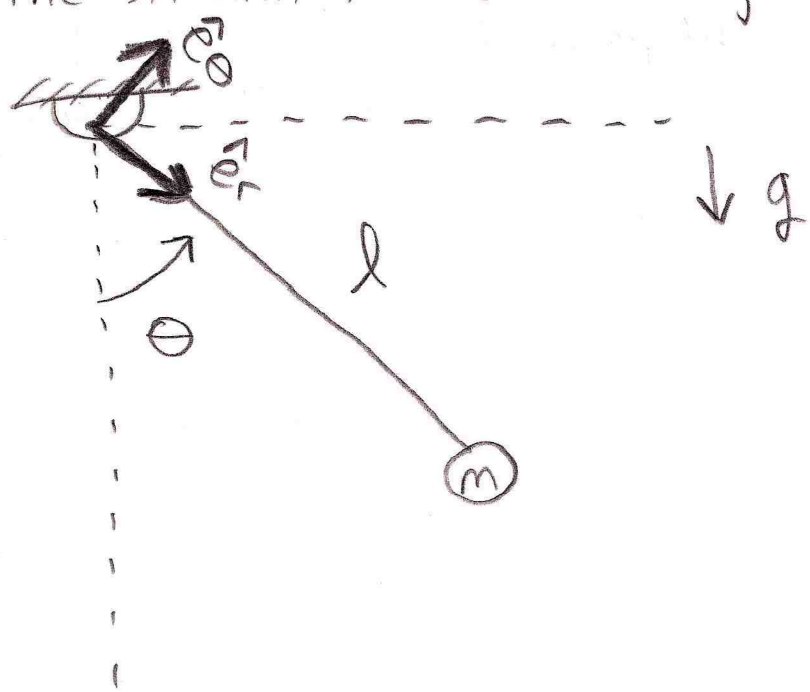
11) Simple pendulum

Derive the simple pendulum equation $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$ as many ways as you can without looking anything up in books. In all cases, use polar coordinates.

The situation is the following:



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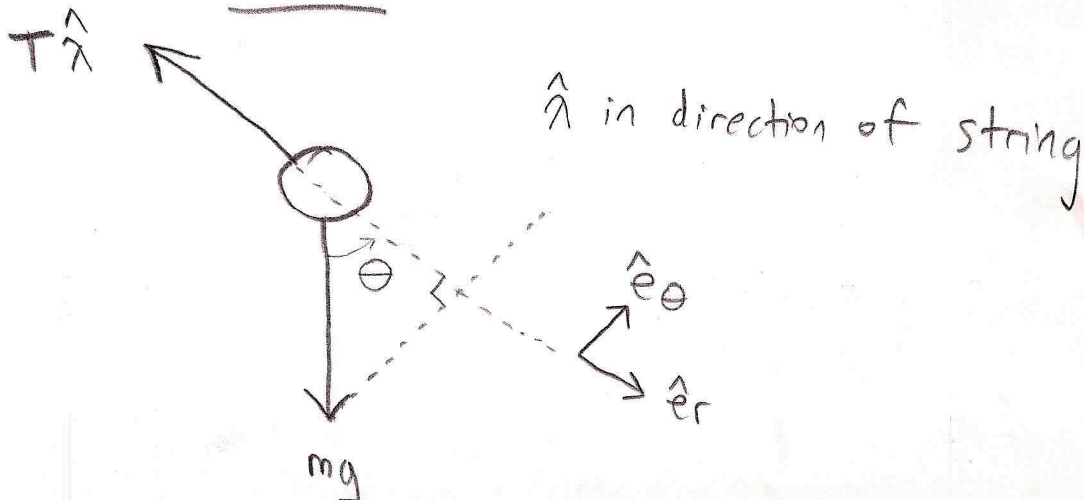


11) (continued)

a) Linear momentum and manipulate the equations to eliminate constraint force.

LMB

$$-T\hat{e}_r + mg\cos\theta\,\hat{e}_r - mg\sin\theta\,\hat{e}_\theta = m\vec{a}$$

For polar coordinates, we have

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta$$

$$\Rightarrow \quad (mg\cos\theta - T)\hat{e}_r - (mg\sin\theta)\hat{e}_\theta = m\left[(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta\right]$$

Constraint Equation

$r = l$

Differentiating twice, we get

$\dot{r} = 0$

$\ddot{r} = 0$

11) (a) (continued)

Component Equations

Dotting the LMB equation with $\hat{e}_r$ gives

$$\{\} \cdot \hat{e}_r \Rightarrow \quad mg\cos\theta - T = m(\ddot{r} - r\dot{\theta}^2)$$

Looking at $\hat{e}_r$ component equation:

$$mg\cos\theta - T = m(\ddot{r} - r\dot{\theta}^2)$$

From constraint equation, we have $\ddot{r} = 0$

$$\Rightarrow \quad mg\cos\theta - T = -mr\dot{\theta}^2$$

$$\Rightarrow \quad T = mg\cos\theta + mr\dot{\theta}^2$$

with $r = l$, we have

$$* \quad T = mg\cos\theta + ml\dot{\theta}^2 \quad *$$

11) (a) (continued)

Plugging this relation for T back into the LMB equation gives

$$\left[mg\cos\theta - (mg\cos\theta + ml\dot{\theta}^2)\right]\hat{e}_r - (mg\sin\theta)\hat{e}_\theta$$

$$\ddot{r}=0 \qquad \dot{r}=0$$

$$= m\left[(\cancel{\ddot{r}} - r\dot{\theta}^2)\hat{e}_r + (\cancel{2\dot{r}\dot{\theta}} + r\ddot{\theta})\hat{e}_\theta\right]$$

$$\Rightarrow (-ml\dot{\theta}^2)\hat{e}_r - (mg\sin\theta)\hat{e}_\theta = (-mr\dot{\theta}^2)\hat{e}_r + (mr\ddot{\theta})\hat{e}_\theta$$

with $r=l$ we have

$$(-ml\dot{\theta}^2)\hat{e}_r - (mg\sin\theta)\hat{e}_\theta = (-ml\dot{\theta}^2)\hat{e}_r + (ml\ddot{\theta})\hat{e}_\theta$$

$$\Rightarrow (-mg\sin\theta)\hat{e}_\theta = (ml\ddot{\theta})\hat{e}_\theta$$

This means that the components must be equal

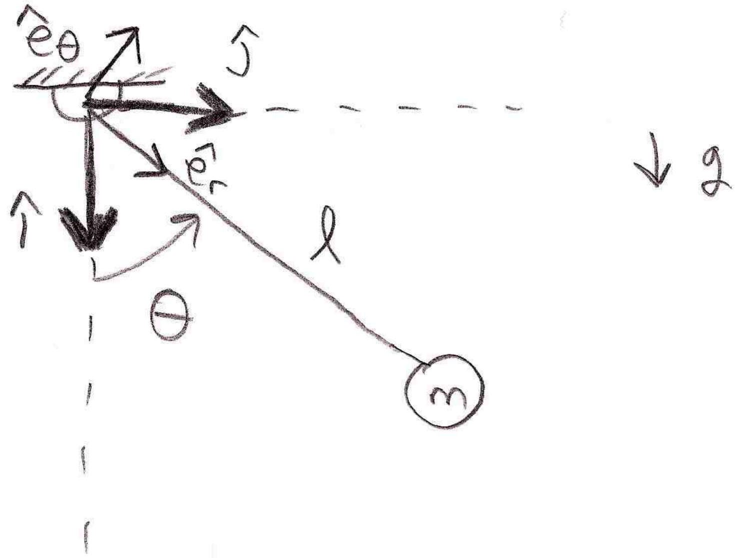
$$\Rightarrow -mg\sin\theta = ml\ddot{\theta}$$

$$\boxed{\ddot{\theta} + \frac{g\sin\theta}{l} = 0}$$

11) a) (continued)

(Not sure if what I just did was what you wanted for manipulating the equations. Since T is only in one direction in polar coordinates it seems like what I did was trivial. I will do the analysis in Cartesian and then convert to polar to show how you can manipulate the equations if T appears in multiple directions)

we have



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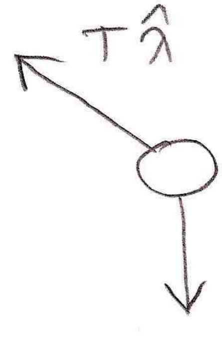


where $\hat{\lambda}$ is in direction of string

11) a) (continued)

LMB

$$mg\hat{\imath} - \frac{Tx}{\sqrt{x^2+y^2}}\hat{\imath} - \frac{Ty}{\sqrt{x^2+y^2}}\hat{\jmath} = m\vec{a}$$

where $\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$

$$\Rightarrow \left(mg - \frac{Tx}{\sqrt{x^2+y^2}}\right)\hat{\imath} - \left(\frac{Ty}{\sqrt{x^2+y^2}}\right)\hat{\jmath} = m\ddot{x}\hat{\imath} + m\ddot{y}\hat{\jmath}$$

Dotting in $\hat{\imath}$ and $\hat{\jmath}$ directions:

$$\{\}\cdot\hat{\imath} \Rightarrow mg - \frac{Tx}{\sqrt{x^2+y^2}} = m\ddot{x} \qquad (Eq.1)$$

$$\{\}\cdot\hat{\jmath} \Rightarrow -\frac{Ty}{\sqrt{x^2+y^2}} = m\ddot{y} \qquad (Eq.2)$$

Multiply Eq.1 by y and Eq.2 by x

$$mgy - \frac{Txy}{\sqrt{x^2+y^2}} = m\ddot{x}y$$

$$-\frac{Txy}{\sqrt{x^2+y^2}} = m\ddot{y}x$$

11) a) (continued)

Subtracting these equations yields

$$mgy - \frac{Txy}{\sqrt{x^2+y^2}} = m\ddot{x}y$$

$$- \qquad \frac{-Txy}{\sqrt{x^2+y^2}} = m\ddot{y}x$$

$$* \quad mgy = m\ddot{x}y - m\ddot{y}x \quad *$$

Now, from the diagram we see that

$$y = l\sin\theta$$

$$x = l\cos\theta$$

$$\Rightarrow \quad \dot{y} = l\dot{\theta}\cos\theta$$

$$\ddot{y} = -l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta$$

$$\dot{x} = -l\dot{\theta}\sin\theta$$

$$\ddot{x} = -l\dot{\theta}^2\cos\theta - l\ddot{\theta}\sin\theta$$

11) a) (continued)

Subbing these relations into the starred equation gives

$$mg(l\sin\theta) = m(-l\dot{\theta}^2\cos\theta - l\ddot{\theta}\sin\theta)(l\sin\theta)$$
$$- m(-l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta)(l\cos\theta)$$

$$mgl\sin\theta = m(-l^2\dot{\theta}^2\cos\theta\sin\theta - l^2\ddot{\theta}\sin^2\theta)$$
$$- m(-l^2\dot{\theta}^2\sin\theta\cos\theta + l^2\ddot{\theta}\cos^2\theta)$$

$$mgl\sin\theta = -ml^2\ddot{\theta}\underbrace{(\sin^2\theta + \cos^2\theta)}_{=1}$$

$$mgl\sin\theta = -ml^2\ddot{\theta}$$

$$\Rightarrow \boxed{\ddot{\theta} + \frac{g\sin\theta}{l} = 0}$$

✓ The long way around the block.

11) b)

Linear Momentum, dot with $\hat{e}_\theta$

From part (a), our LMB equation was

$$(mg\cos\theta - T)\hat{e}_r - (mg\sin\theta)\hat{e}_\theta = m\left[(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta\right]$$

Dotting this with $\hat{e}_\theta$ gives

$$\{\}\cdot\hat{e}_\theta \Rightarrow -mg\sin\theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta})$$

From constraint equations we have that $r = l$ and $\dot{r} = 0$

$$\Rightarrow \quad -mg\sin\theta = m(0 + l\ddot{\theta})$$

$$-mg\sin\theta = ml\ddot{\theta}$$

$$\boxed{\ddot{\theta} + \frac{g\sin\theta}{l} = 0}$$

11) c).

Linear Momentum, cross with $\vec{r}$

we have from LMB

$$(mg\cos\theta - T)\hat{e}_r - (mg\sin\theta)\hat{e}_\theta = m\left[(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta\right]$$

also, $\vec{r} = r\hat{e}_r$

$\Rightarrow$ Crossing the left-hand side of the LMB with $\vec{r}$ gives

$$LHS \times \vec{r} = \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{k} \\ mg\cos\theta - T & -mg\sin\theta & 0 \\ r & 0 & 0 \end{vmatrix}$$

see next page

$$= (mgr\sin\theta)\,\hat{k}$$

I prefer to use the distributive law to using the "determinant" method. Why? Because in probs. like this many terms are zero & you can track the ones that are not zero.

11) c) (continued)

Crossing the right-hand side of the LMB with $\vec{r}$ gives

$$RHS \times \vec{r} = \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{k} \\ m(\ddot{r} - r\dot{\theta}^2) & m(2\dot{r}\dot{\theta} + r\ddot{\theta}) & 0 \\ r & 0 & 0 \end{vmatrix}$$

$$= -mr(2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{k}$$

Equating these two expressions yields

$$\{(mgr\sin\theta)\,\hat{k} = [-mr(2\dot{r}\dot{\theta} + r\ddot{\theta})]\,\hat{k}\}$$

Components must be equal or, $\hat{k} \cdot \{\ \}$

$$\Rightarrow \quad mgr\sin\theta = -mr(2\dot{r}\dot{\theta} + r\ddot{\theta})$$

11) c) (continued)

From the constraint equations we have

$r = l$ and $\dot{r} = 0$

$$\Rightarrow \quad mgl\sin\theta = -ml(0 + l\ddot{\theta})$$

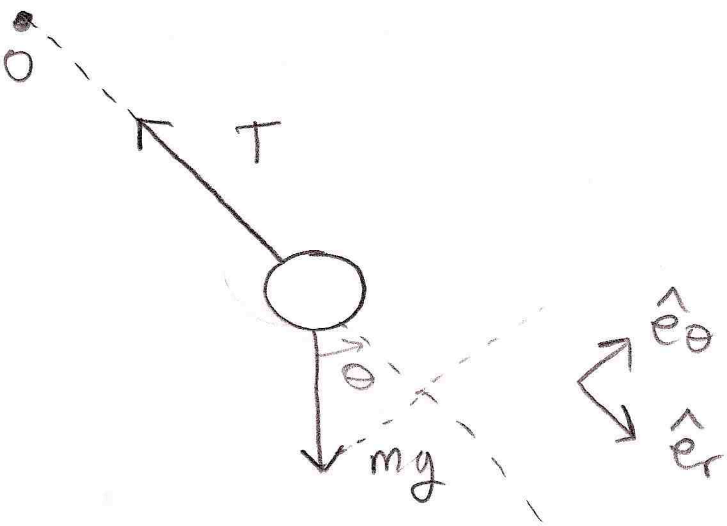
$$mgl\sin\theta = -ml^2\ddot{\theta}$$

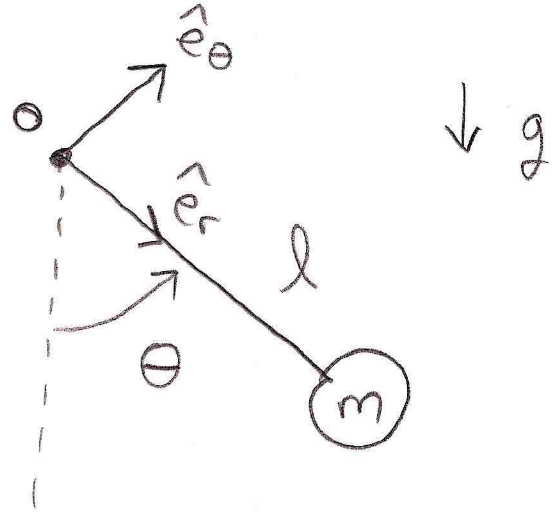
$$\boxed{\ddot{\theta} + \frac{g\sin\theta}{l} = 0}$$

11) d)

Angular Momentum

FBD





AMB/O

$$\sum \vec{M}_{/O} = \dot{\vec{H}}_{/O}$$

$$\left[r\hat{e}_r \times (mg\cos\theta\,\hat{e}_r - mg\sin\theta\,\hat{e}_\theta)\right] + \overbrace{\left[r\hat{e}_r \times (-T\hat{e}_r)\right]}^{0 \text{ since } \hat{e}_r \times \hat{e}_r = 0}$$

$$= r\hat{e}_r \times m\vec{a}$$

11) d) (continued)

0 since $\hat{e}_r \times \hat{e}_r = 0$

$$\Rightarrow \cancel{\left[r\hat{e}_r \times (mg\cos\theta)\hat{e}_r\right]} + \left[r\hat{e}_r \times (-mg\sin\theta\,\hat{e}_\theta)\right] = r\hat{e}_r \times m\vec{a}$$

$$r\hat{e}_r \times (-mg\sin\theta\,\hat{e}_\theta) = r\hat{e}_r \times m\vec{a}$$

Subbing in that $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta$ gives

$$r\hat{e}_r \times (-mg\sin\theta\,\hat{e}_\theta) = \underbrace{r\hat{e}_r \times m[(\ddot{r} - r\dot{\theta}^2)\hat{e}_r}_{\hat{e}_r \times \hat{e}_r = 0} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta]$$

$$\Rightarrow r\hat{e}_r \times (-mg\sin\theta\,\hat{e}_\theta) = r\hat{e}_r \times m(2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{e}_\theta$$

Evaluating the cross product on the left-hand side gives

11) d) (continued)

$$r\hat{e}_r \times (-mg\sin\theta\,\hat{e}_\theta) = \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{k} \\ r & 0 & 0 \\ 0 & -mg\sin\theta & 0 \end{vmatrix}$$

$$= (-mgr\sin\theta)\,\hat{k}$$

Evaluating the cross product on the right-hand side gives

$$r\hat{e}_r \times m(2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta = \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{k} \\ r & 0 & 0 \\ 0 & m(2\dot{r}\dot{\theta} + r\ddot{\theta}) & 0 \end{vmatrix}$$

$$= \left[mr(2\dot{r}\dot{\theta} + r\ddot{\theta})\right]\hat{k}$$

11) d) (continued)

Equating these two expressions gives

$$(-mg\, r\sin\theta)\,\hat{k} = [mr(2\dot{r}\dot{\theta} + r\ddot{\theta})]\,\hat{k}$$

The components must be equal

$$\Rightarrow -mgr\sin\theta = mr(2\dot{r}\dot{\theta} + r\ddot{\theta})$$

Using the constraint relations that $r = l$ and $\dot{r} = 0$ gives

could use earlier to save some calculation

$$-mgl\sin\theta = ml(0 + l\ddot{\theta})$$

$$-mgl\sin\theta = ml^2\ddot{\theta}$$

$$\boxed{\ddot{\theta} + \frac{g\sin\theta}{l} = 0}$$

11) e)

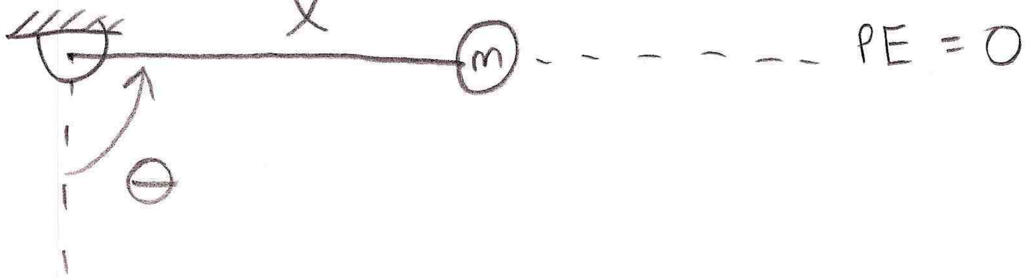
Conservation of Energy

$E_{total} = \text{constant}$

$$\Rightarrow \frac{d}{dt}(E_{total}) = 0$$

If we define the zero potential height at

$\theta = 90°$

$\Rightarrow$ 

Then we can write $E_{tot} = E_K + E_P$

$$E_{tot} = \frac{1}{2} m l^2 \dot{\theta}^2 - m g l \cos\theta$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2} m l^2 \dot{\theta}^2 - m g l \cos\theta\right) = 0$$

11) e) (continued)

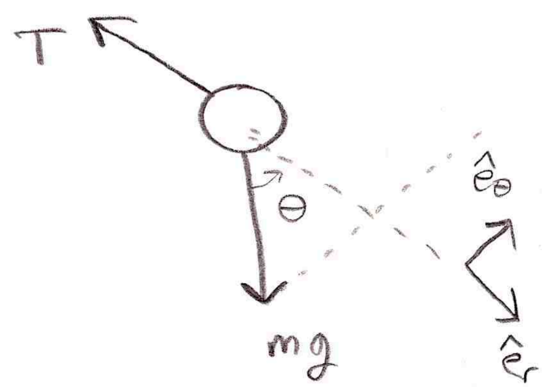
$$\Rightarrow \quad m\ell^2 \dot{\theta}\ddot{\theta} + mg\ell\dot{\theta}\sin\theta = 0$$

$$\boxed{\ddot{\theta} + \frac{g\sin\theta}{\ell} = 0}$$

11) f)

Power Balance

FBD



$$\text{Power} = \dot{E}_K$$

$$\Rightarrow \quad \Sigma \vec{F}_i \cdot \vec{v}_i = \dot{E}_K$$

with $\vec{v} = (\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta)$

$$\vec{F} = (mg\cos\theta - T)\hat{e}_r - (mg\sin\theta)\hat{e}_\theta$$

$$E_K = \frac{1}{2}mr^2\dot{\theta}^2$$

$$\Rightarrow \quad \vec{F}\cdot\vec{v} = \dot{E}_K$$

$$\left[(mg\cos\theta - T)\hat{e}_r - (mg\sin\theta)\hat{e}_\theta\right] \cdot \left[(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta\right]$$

$$= \frac{d}{dt}\left(\frac{1}{2}mr^2\dot{\theta}^2\right)$$

11) f) (continued)

$$(mg\cos\theta)\dot{r} - (mg\sin\theta)r\dot{\theta} = mr^2\dot{\theta}\ddot{\theta}$$

From constraint relations $r = l$ and $\dot{r} = 0$ we get

$$(mg\cos\theta)(0) - (mg\sin\theta)l\dot{\theta} = ml^2\dot{\theta}\ddot{\theta}$$

$$ml^2\dot{\theta}\ddot{\theta} + mgl\dot{\theta}\sin\theta = 0$$

$$\boxed{\ddot{\theta} + \frac{g\sin\theta}{l} = 0}$$

11) g)

Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0$$

where $\mathcal{L} = E_K - E_p$

From the definition of the potential energy given in part (e), we have

$$\mathcal{L} = \frac{1}{2} m l^2 \dot{\theta}^2 - (-mgl\cos\theta)$$

Thus,

$$* \quad \frac{\partial \mathcal{L}}{\partial \theta} = -mgl\sin\theta \quad *$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\Rightarrow * \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} \quad *$$

11) g) (continued)

Thus, we get

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0$$

$$\Rightarrow (-mgl\sin\theta) - (ml^2\ddot{\theta}) = 0$$

$$\boxed{\ddot{\theta} + \frac{g\sin\theta}{l} = 0}$$