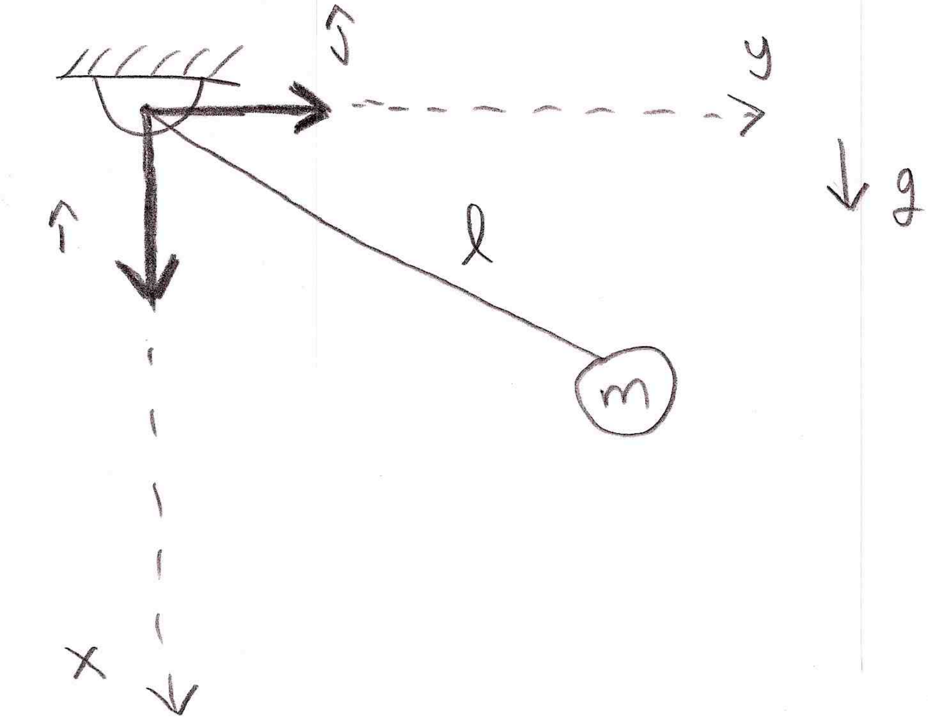
13) <u>Pendulum with an awkward Parameterization</u>

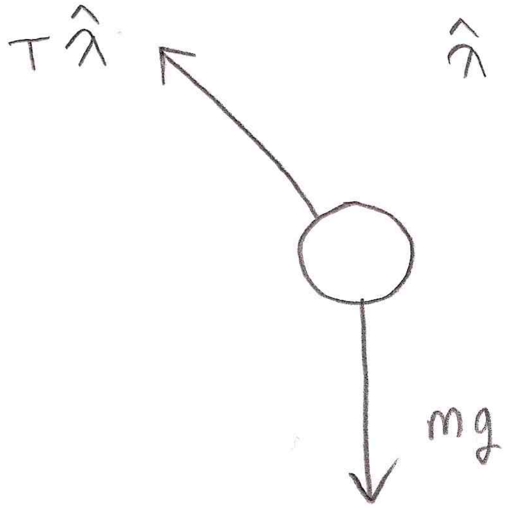
For a simple pendulum, find the equations of motion using y (horizontal position) as your parameterization of the configuration.

We have



13) (continued)

FBD



$\hat{\lambda}$ in direction of string

LMB

$$mg\hat{\imath} + T\hat{\lambda} = m\vec{a}$$

$$\hat{\lambda} = \frac{-x\hat{\imath} - y\hat{\jmath}}{\sqrt{x^2+y^2}} \quad \text{and} \quad \vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}.$$

$$\Rightarrow \left(mg - \frac{Tx}{\sqrt{x^2+y^2}}\right)\hat{\imath} - \left(\frac{Ty}{\sqrt{x^2+y^2}}\right)\hat{\jmath} = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath})$$

13) (continued)

Dotting the LMB equation with $\hat{\imath}$ and $\hat{\jmath}$ gives

$$\{\}\cdot\hat{\imath} \Rightarrow \quad mg - \frac{Tx}{\sqrt{x^2+y^2}} = m\ddot{x} \qquad (\text{Eq. } 1)$$

$$\{\}\cdot\hat{\jmath} \Rightarrow \quad -\frac{Ty}{\sqrt{x^2+y^2}} = m\ddot{y} \qquad (\text{Eq. } 2)$$

Multiplying Eq. (1) by y and Eq. (2) by x and subtracting Eq. (2) from Eq (1) gives

$$mgy - \frac{Txy}{\sqrt{x^2+y^2}} = m\ddot{x}y$$

$$- \quad -\frac{Txy}{\sqrt{x^2+y^2}} = m\ddot{y}x$$

$$* \quad mgy = m\ddot{x}y - m\ddot{y}x \quad *$$

13) (Continued)

Kinematic Constraint Equation

$$x^2 + y^2 = l^2$$

Can solve this equation for x:

$$* \quad x = \sqrt{l^2 - y^2} \quad *$$

Now, let's differentiate the constraint equation once with respect to time:

$$\frac{d}{dt}\left(x^2 + y^2 = l^2\right) \Rightarrow \quad 2x\dot{x} + 2y\dot{y} = 0$$

Can now solve this for $\dot{x}$:

$$\dot{x} = -\frac{y\dot{y}}{x}$$

Subbing in our previous relation for x yields

$$* \quad \dot{x} = \frac{-y\dot{y}}{\sqrt{l^2 - y^2}} \quad *$$

13) (continued)

Now, differentiating the constraint equation a second time with respect to time gives

$$\frac{d^2}{dt^2}\left(x^2+y^2=\ell^2\right) \Rightarrow 2\dot{x}^2+2x\ddot{x}+2\dot{y}^2+2y\ddot{y}=0$$

Can solve this for $\ddot{x}$:

$$\ddot{x} = \frac{-y\ddot{y}-\dot{y}^2-\dot{x}^2}{x}$$

Subbing in for x and $\dot{x}$ yields

$$\ddot{x} = \frac{-y\ddot{y}-\dot{y}^2-\left(\frac{-y\dot{y}}{\sqrt{\ell^2-y^2}}\right)^2}{\sqrt{\ell^2-y^2}}$$

$$* \quad \ddot{x} = \frac{-y\ddot{y}-\dot{y}^2-\frac{y^2\dot{y}^2}{\ell^2-y^2}}{\sqrt{\ell^2-y^2}} \quad *$$

13) (continued)

Now, we had

$$mgy = m\ddot{x}y - m\ddot{y}x$$

Plugging in our expressions for x and $\ddot{x}$ yields

$$gy = y\left(\frac{-y\ddot{y} - \dot{y}^2 - \frac{y^2\dot{y}^2}{l^2-y^2}}{\sqrt{l^2-y^2}}\right) - \ddot{y}\left(\sqrt{l^2-y^2}\right)$$

$$gy = \left[\frac{-y^2\ddot{y} - y\dot{y}^2 - \frac{y^3\dot{y}^2}{l^2-y^2}}{\sqrt{l^2-y^2}}\right] - \ddot{y}\sqrt{l^2-y^2}$$

$$gy = \frac{\left[\frac{-l^2y^2\ddot{y} + y^4\ddot{y} - l^2y\dot{y}^2 + y^3\dot{y}^2 - y^3\dot{y}^2}{l^2-y^2}\right]}{\sqrt{l^2-y^2}} - \ddot{y}\sqrt{l^2-y^2}$$

13) (continued)

$$gy(l^2-y^2)\sqrt{l^2-y^2} = -l^2y^2\ddot{y} + y^4\ddot{y} - l^2y\dot{y}^2 - \ddot{y}(l^2-y^2)^2$$

$$gy(l^2-y^2)\sqrt{l^2-y^2} = -l^2y^2\ddot{y} + y^4\ddot{y} - l^2y\dot{y}^2 - \ddot{y}(l^4-2l^2y^2+y^4)$$

$$gy(l^2-y^2)\sqrt{l^2-y^2} = -l^2y^2\ddot{y} + y^4\ddot{y} - l^2y\dot{y}^2 - l^4\ddot{y} + 2l^2y^2\ddot{y} - y^4\ddot{y}$$

$$gy(l^2-y^2)\sqrt{l^2-y^2} = l^2y^2\ddot{y} - l^4\ddot{y} - l^2y\dot{y}^2$$

$$gy(l^2-y^2)\sqrt{l^2-y^2} = \ddot{y}(l^2y^2-l^4) - l^2y\dot{y}^2$$

$$\Rightarrow \boxed{\ddot{y} = \frac{gy(l^2-y^2)\sqrt{l^2-y^2} + \dot{y}^2(l^2y)}{l^2y^2-l^4}}$$

Wow!

13) (continued)

The solution to this 2nd order ODE was computed using Matlab (see attached code and plots).

I let $m = 1 kg$

$\ell = 1 m$

$g = 1 m/s^2$

$y_0 = 0 \ m$

$\dot{y}_0 = 1 \ m/s$

The comparison between this solution and the solution to the simple pendulum equation is also shown. In order to accurately compare, we must determine the initial $\dot{\theta}_0$ that corresponds to $\dot{y}_0 = 1 m/s$

13) (continued)

we know that if $y_0 = 0$ m, then $\underline{\theta_0 = 0}$ rad.

Also, $y = l \sin\theta$

$$\Rightarrow \quad \dot{y} = l\dot{\theta}\cos\theta$$

$$\Rightarrow \quad \dot{\theta} = \frac{\dot{y}}{l\cos\theta}$$

$$\dot{\theta}_0 = \frac{\dot{y}_0}{l\cos(\theta_0)}$$

If $\theta_0 = 0$ radians, $\dot{y}_0 = 1$ m/s, and $l = 1$ m we get

$$\dot{\theta}_0 = \frac{(1\ \text{m/s})}{(1\text{m})\cos(0)}$$

$$* \Rightarrow \quad \dot{\theta}_0 = 1\ \text{rad/s} \quad *$$

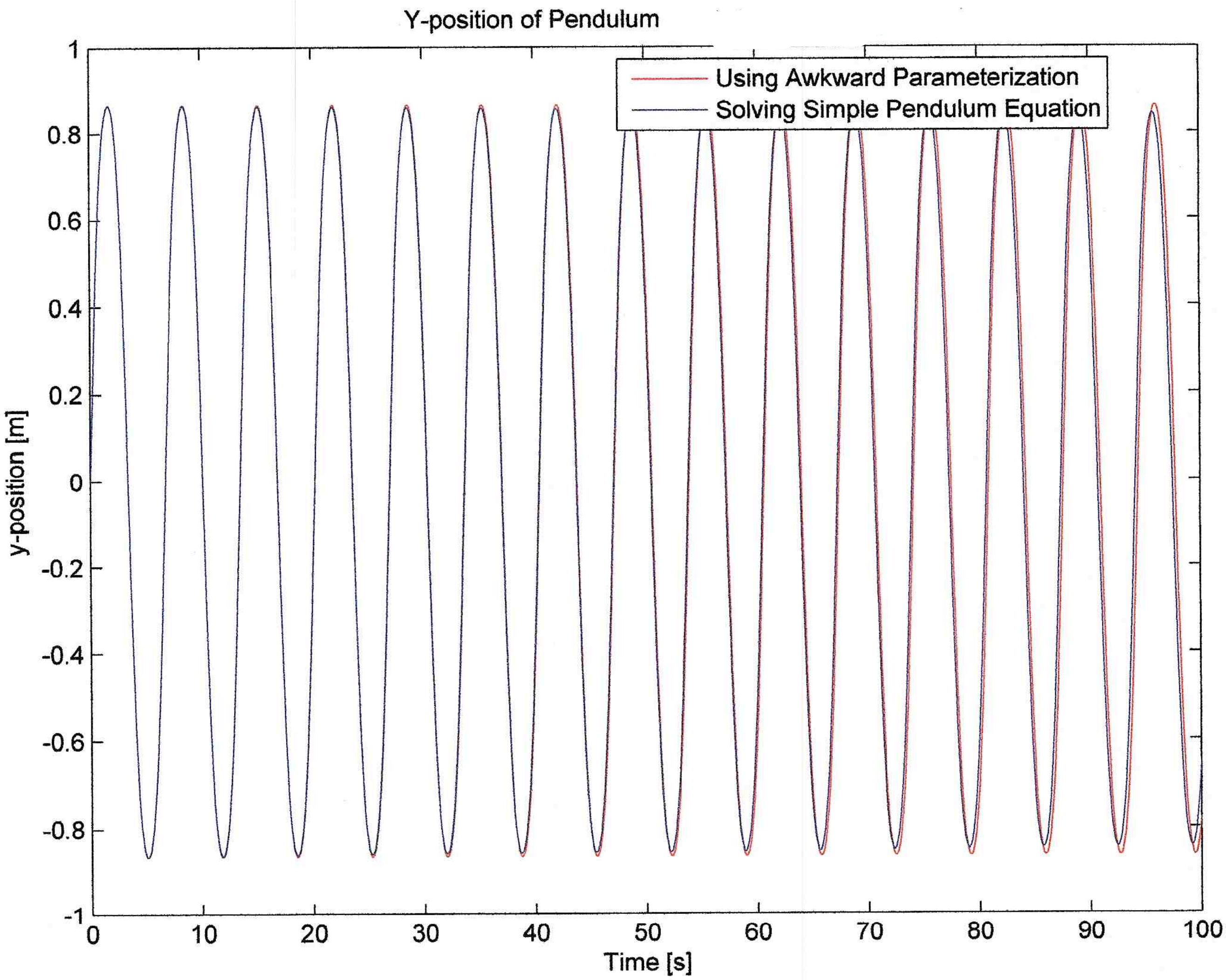
* See attached code and plots *

```
%Homework 5, Problem 13

function MAE5735_ParameterizedPendulum()
%This function calculated the motion of a simple pendulum that is
%parameterized in the variable y as well as the motion using the simple
%pendulum equation. These two methods are then compared.

clc
clear all
close all

%The following are pertinent constants for this scenario.
p.m = 1; %Mass of the pendulum in kg
p.L = 1; %Length of the pendulum in m
p.g = 1; %Gravitational acceleration in m/s^2

%The following establishes the time span and initial conditions for the
%ODE solution.

tspan = linspace(0,100,1001); %Timespan under consideration

%Initial conditions for y-parameterization
y0 = 0; %Initial y-position in m. The pendulum is initially hanging straight down.
y_dot0 = 1; %Initial y-velocity vector in m/s. Initially the pendulum begins moving
            %in the y-direction with a speed of 2 m/s
z0_y = [y0;y_dot0]; %This vector contains the position and velocity information
                    %of the pendulum in the y-direction to be solved by the
                    %ODE solver.

%Initial conditions for simple pendulum equation
theta0 = 0; %Theta starts are zero radians, i.e., the pendulum is hanging straight
down
theta_dot0 = 1; %Initial angular velocity is 1 rad/s
z0_theta = [theta0;theta_dot0]; %Vector containing intial angle and angular velocity
information

%The following solves the differential equation outlined in the subfunction
%"zdot_y" below for the y-parameterization equation
[t z_y] = ode23(@rhs_parameterized,tspan,z0_y,[],p);

%Isolate the y-position of the pendulum.
y_parameterized = z_y(:,1);

%The following solves the differential equation outlined in the subfunction
%"zdot_theta" below for the simple pendulum equation
[t z_theta] = ode23(@rhs_theta,tspan,z0_theta,[],p);

%Isolate the angle of the pendulum.
```

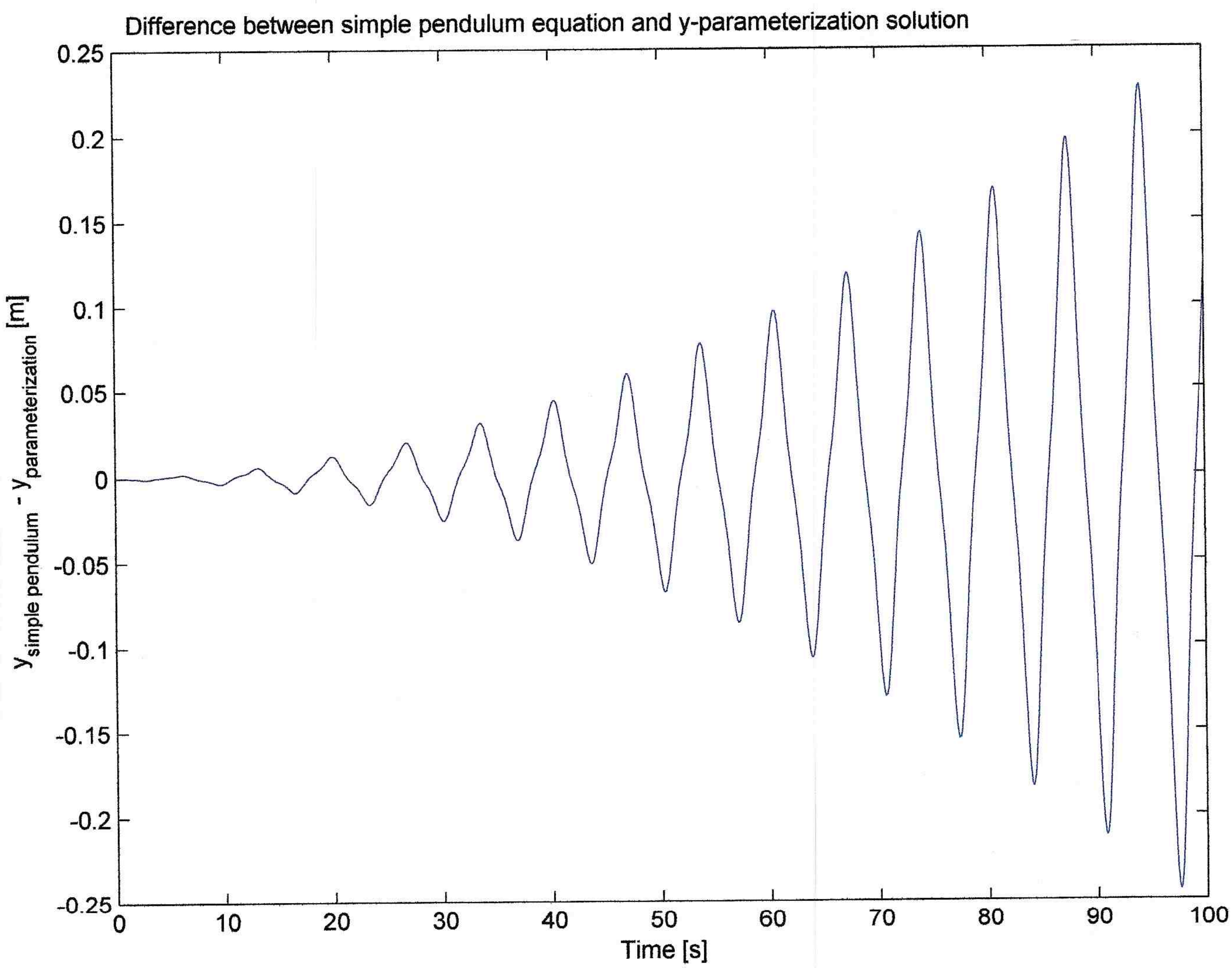
```
theta = z_theta(:,1);

%We can now compute the y-position from the angle information since
%y = l*sin(theta)
y_simplependulum = (p.L).*sin(theta);

%The following plots the motion of the pendulum in the y-direction for
%both methods
plot(t,y_parameterized,'r',t,y_simplependulum,'b')
xlabel('Time [s]')
ylabel('y-position [m]')
title('Y-position of Pendulum                    ')
legend('Using Awkward Parameterization','Solving Simple Pendulum Equation')

figure

%The following plots the difference between the simple pendulum equation
%solution and the y-parameterization solution
plot(t,y_simplependulum-y_parameterized,'b')
xlabel('Time [s]')
ylabel('y_s_i_m_p_l_e_ _p_e_n_d_u_l_u_m - y_p_a_r_a_m_e_t_e_r_i_z_a_t_i_o_n [m]')
title('Difference between simple pendulum equation and y-parameterization solution
               ')


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%THE FOLLOWING ARE SUBFUNCTIONS UTILIZED BY THIS CODE%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function zdot_y =rhs_parameterized(t,z,p)
%This function calculates the time derivative of a given input vector.
y = z(1); %The y-position of the pendulum
y_dot = z(2); %The y-velocity of the pendulum

%The following evaluates the second derivative of y based on the equation
%derived in the homework.
y_doubledot = (p.g*y*((p.L)^2-y^2)*sqrt((p.L)^2-y^2)+(y_dot)^2*((p.L)^2*y))/((y^2)*(p.
L)^2-(p.L)^4);

zdot_y = [y_dot;y_doubledot]; %Collects the important parameters, i.e., the time
derivatives
                              %of the input quantities.
end

function zdot_theta =rhs_theta(t,z,p)
%This function calculates the time derivative of a given input vector.
theta = z(1); %The angle of the pendulum
theta_dot = z(2); %The angular velocity of the pendulum

%The following evaluates the second derivative of theta based on the simple
```

```
%pendulum equation
theta_doubledot = (-(p.g)/(p.L))*sin(theta);

zdot_theta = [theta_dot;theta_doubledot]; %Collects the important parameters, i.e.,
the time derivatives
                                          %of the input quantities.

end

end
```

Really Good



Y-position of Pendulum
Using Awkward Parameterization
Solving Simple Pendulum Equation
y-position [m]
Time [s]
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
10
20
30
40
50
60
70
80
90
100


Difference between simple pendulum equation and y-parameterization solution
0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
0
10
20
30
40
50
60
70
80
90
100
Time [s]
$y_{simple\ pendulum} - y_{parameterization}$ [m]

13) (continued)

From the previous plots, it is clear that the y-parameterization solution agrees reasonably well with the solution from the simple pendulum equation for small time spans. However, the longer that the simulation is run, the more pronounced the differences between the two methods become (this is seen in the plot of $y_{\text{simple pendulum}} - y_{\text{parameterization}}$).